# STUDIES ON FATIGUE OF METALS ON THE BASIS OF A STOCHASTIC MODEL FOR DISLOCATION MULTIPLICATION

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# ABSTRACT


Fatigue of metals is studied on the basis of a stochastic model for damage accumulation. In this model the fatigue damage accumulates inside the materials as a form of dislocation multiplied randomly.

First, crack initiation in high-cycle fatigue is investigated. A stochastic model for crack initiation based on accumulation of fatigue damage is proposed, and fatigue life up to cell-sized crack initiation is estimated. Cell size is derived by a probabilistic method, and by assumptions on the growing process from a cell-sized crack to a grain-sized one, the life up to grain-sized crack initiation is calculated. The results are compared with the experimental data for S20C steel.

Secondly, small fatigue crack propagation, which seems to be quite different from through crack propagation, is investigated. The modified J-integral $J^{(N)}$ (fatigue J-integral), which is calculated on the basis of the dislocation multiplication model, is introduced. It is shown that the propagation rate of small fatigue cracks is proportional to $J^{(N)}$. Moreover, using the propagation law of small fatigue cracks, the fatigue life up to fracture is estimated.

n some respects in order that elasto-plastic fa- omena can be treated. With the use of the odel, elasto-plastic fatigue is studied.

ly, a model for the propagation of fatigue through ich is a kind of weakest link model, is proposed. med that the region near the crack tip d of small material elements and that the life of nt is a random variable. When an element crack is assumed to propagate to the element. model, it is shown that the fatigue crack propaga- is proportional to some power of the stress intensity Furthermore the analytic result is compared xperimental data, using the distribution func- fe of each element which is obtained from the umulation model.

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CONTENTS

# CHAPTER 1

# INTRODUCTION

Fatigue is one type of material failure. If proper precautions are paid to creep and corrosion, a structure subjected to a steady and static force less than the ultimate strength of the metal should never break. On the other hand, if a structure is subjected to a cyclic, repeated or fluctuating load, it may fracture at a stress level lower than that required to cause failure under static conditions. This phenomenon is known as fatigue which is a common source of primary failure of metals. In this article, the fatigue phenomenon is investigated on the basis of a stochastic model for dislocation multiplications.

First, we make a historical survey of studies on fatigue. This fatigue phenomenon started to worry engineers more than a hundred years ago. In 1830 Albert[1] attempted to load repeated stress on welded mine hoist chains, continuing some tests up to $10^5$ cycles. In 1840 Rankine[2] reported that materials suddenly broke with no gross deformation under repeated loadings. Between 1850 and 1865, both Hodgkinson[3] and Fairbairn[4] carried out repeated bending tests on beams. Fairbairn used a mechanism actuated by a water-wheel to apply repeatedly a load to the center of a 6.7m long wrought iron built-up girder. He found that the beam did not break under a static load of less than

120kN, yet nevertheless broke by repeated loads of only 30kN.

Because these failures occurred in a material that had functioned satisfactorily for a certain time, the general opinion developed that the material had tired of carrying the load or that the continual reapplication of a load had in some way exhausted the ability of the material to carry load. Thus the word "fatigue" was coined to describe such failures, and the name has survived to this day.

For a long time, engineers and metallurgists have sought the reason for the failure of materials under the repeated application of a stress which, if applied once, would not cause any failures. Wöhler [5] constructed various types of fatigue testing machines and systematically carried out the first fatigue tests on metallic specimens,paying special attention to the magnitude of the applied loads. From tests on iron and steel specimens, he found that the range of applied stress rather than the maximum tensile stress in the loading cycle determined the life of a specimen, and that there existed stress amplitude below which a specimen did not break no matter how many times the stress cycle was applied. Since Wöhler's work, various materials have been tested under different conditions of loading and environment in order to investigate the fatigue properties[6].

From these experimental results, it is considered that fatigue fracture has the following properties.

(1) Even for stress amplitude $S$ less than static strength or elastic limit, fatigue failure occurs under the repeated application of stress.

(2) Until fracture, macroscopic deformation does not occur in materials.

(3) The average cycle number of repeated stress up to fracture is a function of $S$. The relation between $S$ and average cycle number or simply life $N$, that is, S-N curve or Wöhler's curve, is independent of the frequency of repeated stress.

(4) For steel and other metals which has the b.c.c. structure, there exists a specific value of stress amplitude below wnich specimens do not fail. It is called the endurance limit.

Recently,electron microscopy, X-ray diffraction technique, acoustic emission and many other new techniques have been applied to study the microscopic structure of materials under repeated stress. Information obtained by these techniques has been accumulated and has given light on the micro - mechanism of fatigue.

In the first place we explain that the fatigue process is divided roughly into two stages.

The general and basic features of fatigue failure have been recognized as follows. When an external cyclic stress is applied to a material, dislocation multiplications take place, and slip bands are formed locally on the surface where slip occurs easily. At the same time, work hardening proceeds in the material, and microstructure named cell or subgrain develops. Hereafter we call it cell for simplicity.

After some load cycles, work-hardening is saturated. The number of load cycle spent before the saturation of work-hardening is much less than total fatigue life.

In a surface layer where dislocations pile up microcracks soon occur. Subsequently they extend through the layer and penetrate into the body of the metal If a surface layer is removed at frequent intervals throughout a test, fatigue life increases irrespective of stress amplitude[7]. This fact shows that crack initiation is restricted to the surface grains.

The surface microcrack initially propagates in the direction parallel to the operative slip planes. This means that the crack growth process is affected by microscopic crystalline structure of materials. Hereafter, this microcrack will be referred to as Stage I crack[8].

Stage I continues until a microcrack becomes so large that it seems to grow independently of the microstructure of materials. For a crack of such a size,which is of the order of grain size, the scale of the plastic region around the crack tip exceeds several times the grain size. Therefore, the crack may be considered to propagate in a continuum and its propagation is affected largely by the tensile strain near the crack edge. The crack now grows independently of the microstructure. The crack propagation, relating to the magnitude of the tensile strain near the crack tip, is restricted within a plane which is approximately perpendicular to the applied cyclic stress. This plane has no relation with any crystalline slip planes. A crack at this stage is macroscopic and can be termed a macrocrack. It will be called Stage II crack[8] in what follows. Its growth rate is much larger than that of the initial microcrack, and it now spreads rapidly through the metal. Eventually it grows so large that the material fails. Following Stage I, crack propagation process up to fracture is named Stage II.

Next, we give an explanation of some characteristics of the two stages.

First, in Stage I, an important factor of microscopic crack initiation is associated with slip line and cell formation. Development of slip-line on the surface of a fatigue specimen has been studied by many workers since the turn of the century. In 1903, Ewing and Humphrey[9] tested specimens of Swedish iron in rotating bending at stress levels above their fatigue limit. In the experiment a test was stopped at frequent intervals and specimen surface was polished and etched. They observed the following: initially few slip lines formed, and as a test proceeded, new slip lines formed closely to them, producing bands of slip. Although there existed regions between the bands where slip-lines were not formed, these bands grew wider and denser. Fatigue cracks eventually formed in the broadened bands.

The results of Ewing and Humphrey were reconfirmed by successive workers who observed development of a microcrack in ductile polycrystals under cyclic stress with metallographic techniques. For example, Thompson, Wadsworth and Louat[10] tested annealed, electropolished, polycrystalline high-purity copper specimens in reversed stress (zero mean load), and found the followings. Slip bands began to appear early in a test and became more and more numerous as the test proceeded. Most of the slip bands were removed by electropolishing, and became invisible. A few slip bands, however, could not be removed and hence were termed "persistent slip bands". Fatigue cracks grew eventually from these bands. In some tests, persistent slip bands were observed after only 5 per cent of the total life, but they did not in general extend far down into the body of the metal.

Next we give a description of Stage II. We can divide Stage II moreover into three stages, Stage $II_a$, $II_b$[11] and $II_c$[12]. In stage $II_a$ there still remains the microstructural effect on cracking, and in Stage $II_b$ crack propagation can be treated by fracture mechanics. Stage $II_c$ is the last stage containing unstable fracture. Among these stages, Stage $II_b$ is investigated most extensively. Therefore, properties of crack growth in Stage $II_b$ are described, first.

In Stage $II_b$ microstructure does not affect the crack growth process. Paris and his co-workers[13], [14] showed that the crack propagation rate $da/dn$ in Stage $II_b$ can be expressed as a function of fracture mechanical parameter, i.e. range of stress intensity factor $\Delta K$,

$$\frac{da}{dn} = f(\Delta K), \qquad (1\text{-}1)$$

where $\Delta K$ is defined by

$$\Delta K = (S_{max} - S_{min})\sqrt{\pi a}\ ,$$

$a$ is the crack length, and $S_{max}$ and $S_{min}$ are the maximum and minimum stress in one cycle. Equation (1-1) is well confirmed by many workers[15].

In the case of elastic stress field the stress intensity factor is a parameter sufficient to describe the stress field near the tip of a crack. When the size of a plastic region at the crack tip is small compared to the crack length, the stress intensity factor may still give a good indication of the stress environment of the crack tip. If two different cracks have the same stress environment, i.e. the same stress intensity factor, they behave in the same manner and show equal rates of growth.

The double-logarithmic plot of $da/dn$ versus $\Delta K$ has an S-shaped curve, or at least consists of parts with different slopes[16], [17]. For a suitable range of $\Delta K$, however, $da/dn$ is proportional to the power of $\Delta K$, i.e.

$$\frac{da}{dn} = C(\Delta K)^m, \qquad (1\text{-}2)$$

where $C$ and $m$ are constants. The $m$ usually takes values between $2$ and $4$. Near the upper end of the $\Delta K$ range a large deviation from equation (1-2) appears where the crack is reaching a critical size and hence where $da/dn$ becomes infinitely large. Then, the final fracture occurs during the cycle in which the stress intensity reaches a certain value. At this time Stage $II_b$ ends and Stage $II_c$ begins. How to treat the final fracture is left unknown.

In Stage $II_a$, however, it is reported that the growth rate of cracks cannot be expressed as a function of $\Delta K$ as in Stage $II_b$ [18],[19],[20],[21]. Many workers being concerned with the propagation of small cracks, a number of phenomenological laws of fatigue crack propagation in Stage $II_a$ have been proposed. It is proposed that in Stage $II_a$ the crack propagation rate can be expressed as $\Delta K_{eff}$ (effective range of stress intensity factor) [20],[21] or that it is proportional to some power of $\Delta J$ [19], i.e. range of J-integral whose definition and meaning are shown in chapter 3. Kitagawa et.al.[18] reported that the growth rate can be regarded as a function of the strain intensity factor. In spite of these works, an appropriate treatment of the early stage of the crack propagation process is still not available.

Until now, we have described the "average" properties of fatigue, i.e. S-N curves and equation (1-2) are expressed by average values. Empirically, fatigue lives are distributed over a wide range and their deviations are generally inevitable because of the heterogeneity which exists randomly in materials. Therefore, paying attention to this point, Yokobori and his coworkers developed a stochastic theory of fatigue of metals[22].

On the basis of the foregoing facts, we want to describe fatigue phenomena as follows.

When an external cyclic stress is applied to a material, microstructure (cell) develops. We determine the size of this structure by probabilistic method.

After some load cycles, work-hardening is saturated. We pay attention only to the period subsequent to this. This period covers the greater part of the fatigue life. Here, dislocations make flip-flop motion in almost all load cycles, and their density scarcely increases. On a rare occasion, however, dislocation sources (for instance Frank-Read source) are activated and dislocation multiplications occur. We relate accumulation of fatigue damage to the multiplication and piling-up of dislocations in materials. That is to say, we regard strain energy due to the dislocations multiplied as a measure of fatigue damage. Hereafter we refer to strain energy as damage, for simplicity. Furthermore, we treat this accumulative process as a random one because of the heterogenity of materials.

We propose that a microscopic crack initiates near the surface of a grain when the accumulated damage exceeds a specific value. The size of the initiated crack may be of the order of cell size, since the crack occurs due to piling-up of dislocations.

The cell-sized crack grows under cyclic stress, and its size attains grain size after some load cycles. We regard this growing stage as Stage I.

We treat the crack propagation process in Stage I as follows. When a cell-sized crack crosses a cell, the fatigue damage begins to accumulate in the adjacent cell and some further cycles are spent before the crack can cross the cell; then the crack propagates across this cell. This process is repeated until the crack length is equal to the grain size. From these considerations, we derive the expression for fatigue life up to grain-sized crack initiation, i.e. the end of Stage I, and compare it with experiments.

In Stage $II_a$, following Stage I, the crack propagates on the plane normal to the maximum principal tensile stress. We ascertain that the growth rate of Stage $II_a$ crack is proportional to fatigue J-integral which is proposed in chapter 3.

In the next stage, i.e. Stage $II_b$, the crack propagation process is called subcritical growth, namely the growth rate is expressed by equation (1-2) which is well established for the growth of through cracks. We derive equation (1-2) with the use of a kind of weakest link model.

The crack continues to propagate subcritically until unstable fracture (Stage $II_c$) occurs, and the material finally breaks.

We summarize the contexts of the following chapters below.

In Chapter 2 we propose a stochastic model for the fatigue of metals under repeated stress. Fatigue life of metals up to crack initiation is investigated with the stochastic model based on accumulation of fatigue damage. The accumulation process is assumed to be generated by a Poisson process. Some parameters adopted in this model are related to the material properties obtained from the static stress-strain tension test, and our theoretical results are compared with the experiments using three kinds of low-carbon steel S20C with different grain sizes. The applicability of our model to fatigue crack initiation is examined. The context of this chapter is taken from a published paper [23].

In chapter 3 the growth law of small fatigue cracks is also investigated on the basis of a dislocation multiplication model. A new growth law of small fatigue cracks is proposed on the analogy of the growth law of small creep cracks. The modified J-integral $J^{(N)}$ (fatigue J-integral) which corresponds to the creep J-integral $J'$ in the case of creep is defined, and it is shown that the growth rate of small fatigue cracks is proportional to $J^{(N)}$. With the crack initiation model proposed in chapter 2 and this growth law of small fatigue cracks, total fatigue life is evaluated. The context of chapter 3 is also taken from a published paper[24].

In chapter 4 the models proposed in chapter 2 and 3 are improved in order that elasto-plastic fatigue phenomena can be treated. In elasto-plastic fatigue, plastic deformation must be taken into account and damage accumulation process cannot be assumed to be generated by a Poisson process because of frequant occurrence of damage accumulation. With the model improved in this respect, fatigue life up to crack initiation is calculated and the results are compared

with experimental data. Furthermore, the fatigue J-integral proposed in chapter 3 is obtained with the use of this model and it is shown to be proportional to the growth rate of small cracks.
With this crack initiation model and the growth law of small fatigue crack, fatigue life up to failure is estimated. The context of chapter 4 is taken from reference 25.

In chapter 5 a probabilistic model for fatigue crack growth proposed by Oh[26] is modified in some respects under more natural assumptions than Oh's. It is shown that the rate of fatigue crack growth is proportional to some power of the stress intensity factor. It is also shown that the exponent ranges from 2 to 4.
Furthermore, the propagation rate of the through fatigue crack is investigated on the basis of a dislocation multiplication model which is used in evaluating the fatigue life up to crack initiation in previous chapters. The expression for the fatigue crack propagation rate is derived from a two-dimensional version of our probabilistic model for fatigue crack growth proposed above and a distribution function of the life of material elements in this expression is calculated on the dislocation multiplication model.
On this basis, it is also shown that the fatigue crack propagation rate is proportional to some power of the

stress intensity range. Our results are compared with the experimental data. The context of chapter 5 is taken from published papers [27],[28] and [29].

Finally, in chapter 6, we state our concluding remarks and make some remarks on the perspective of our study on fatigue of metals.

# CHAPTER 2

# CRACK INITIATION IN HIGH CYCLE FATIGUE

## 2-1 Introduction

The dominant feature of fatigue phenomena comprises local plastic strain, cyclic hardening and softening, and crack initiation and its growth. In general, the greater part of fatigue life is expended on crack initiation and its propagation process. Recently many workers have focussed their attention on the initiation and growing process of cracks, that is, they have investigated the propagation rate of small fatigue cracks and lives up to crack initiation[30],[31]. In this chapter we direct our attention to crack initiation.

We assume that the mechanism of fatigue crack initiation is as follows. Under repeated stress, many dislocation sources which have adequate orientations to the applied stress begin to multiply dislocation loops in materials. The dislocations cause local accumulation of strain energy inside the materials. We regard the strain energy as fatigue damage. Since in practice the same stress cannot be always applied to materials in each cycle, response of materials in each cycle is irregular. Therefore, temporal occurance of dislocation multiplications is random, and hence the accumulation process of fatigue damage is a random one. We assume that when the accumulated fatigue damage exceeds a specific value, microcrack initiates. Since the crack occurs due to piling-up of dislocations, the size of the crack is of the order of cell size. The cell-sized crack initiated near the surface of materials grows until its length attains the grain size. This crack growth can be considered as Stage I crack growth.

In this chapter, we give a stochastic model for the fatigue crack initiation in the saturation hardening stage that covers the greater part of the fatigue life. This model is based on the cumulative damage model proposed by Ihara and Tsurui[32]. In this model, the average size of the cell is determined by a probabilistic method, and the propagation process in Stage I is treated under suitable assumptions. Using this model, we obtain mean life up to fatigue crack initiation. Our results are compared with the three kinds of low-carbon steel S20C with different grain sizes[31] and the applicability of our model to fatigue crack initiation is examined.

In section 2-2, our stochastic model for fatigue crack initiation is explained and a expression of fatigue life up to grain-sized crack is derived. In section 2-3, the cell formation is discussed and cell size is calculated. In section 2-4, some parameters used in the model are related to the material property obtained from the static stress-strain tension test. Our results are compared with experiments in section 2-5. Section 2-6 contains some extra remarks.

## 2-2 A Stochastic Model for Crack Initiation

Suppose that external cyclic stress $S\sin(\omega t)$ is loaded on materials (where $S$ denotes stress amplitude and $\omega$ the angular frequancy), then strain energy is stored transiently inside the materials. The strain energy per unit cycle, $Y$, is considered to be a random variable with exponential distribution

$$\Pr.\{Y \leqq y\} = H(y) = \begin{cases} 1 - \exp(-\frac{y}{\sigma}) & y \geqq 0 \\ 0 & , y < 0 \end{cases} \qquad (2\text{-}1)$$

where the mean $\sigma$ is assumed to be a constant. We assume that a part of the transiently stored energy less than a specified value $Q$ is expended as kinetic or thermal energy and hence has no direct effect on the formation of dislocations, and that, if $Y>Q$, the remainder of the energy, $Y-Q$, is only accumulated as deformation energy in the form of dislocations. This accumulated part is regarded as a so called fatigue damage which causes fatigue failure. Hereafter the event $Y>Q$ is referred to as $\mathcal{A}$. Since the probability that the event $\mathcal{A}$ occurs in one cycle is

$$\Pr.\{\mathcal{A}\}=\Pr.\{Y > Q\} = \exp(-\frac{Q}{\sigma}),$$

the average time interval $1/\lambda$ that succesive $\mathcal{A}$'s occur becomes

$$\frac{1}{\lambda} = \frac{2\pi}{\omega} \exp(\frac{Q}{\sigma}). \qquad (2\text{-}2)$$

If we put $X=Y-Q$, the conditional random variable $X$ subject to the restriction that $\mathcal{A}$ must occur has the same distribution as equation (2-1), i.e. $H(x)$.

Since the event $\mathcal{A}$ occurs randomly, moreover rarely, we may assume, for simplicity, that $\mathcal{A}$ occurs according to a Poisson process with parameter $\lambda$. We note that even the accumulated part decays very slowly with decay parameter $\alpha$ under the repeated stress, as there are occasions when dislocations annihilate each other under such circumstances. Therefore, though both accumulation and decay of the damage occur, the damage will accumulate gradually.

Furthermore, it is assumed that a cell-sized crack occurs when the accumulated part of strain energy exceeds a specified value $U$ for the first time. Then, with the aid of the foregoing assumptions, the average time $\overline{T}$ to cell-sized crack initiation can be explicitly obtained for the integral values of $\lambda/\alpha$. Let us express its result as follows:

$$\bar{T} = \frac{1}{\lambda} F_l \left(\frac{U}{\sigma}\right), \tag{2-3}$$

where

$$l = \frac{\lambda}{\alpha}. \tag{2-4}$$

The explicit expression of $F_l(x)$ and the derivation of equation (2-3) for the integral values of $l$ are given in Appendix 2-1. When $l$ is not an integer, $F_l(x)$ can be obtained by interpolation. It must be noted that the parameter $l$ in equation (2-3) which denotes the ratio $\lambda/\alpha$ also depends on $\sigma$ through the relation (2-2).

Making use of equation (2-3), we get the mean fatigue life $\bar{N}_{ci}$ (measured by cycle number) up to cell-sized crack initiation as follows:

$$\bar{N}_{ci} = \frac{\omega}{2\pi} \bar{T}. \tag{2-5}$$

Equations (2-2), (2-3) and (2-5) lead to the final result

$$\overline{N}_{ci} = \exp\left(\frac{Q}{\sigma}\right) F_l\left(\frac{U}{\sigma}\right), \qquad (2\text{-}6)$$

or

$$\log\overline{N}_{ci} = \frac{Q}{\sigma}\log e + \log F_l\left(\frac{U}{\sigma}\right). \qquad (2\text{-}7)$$

Here, we specify the expression of $\alpha$. Under the repeated stress, dislocation segments are in shuttling motion with friction[33] and the average velocity of that motion may be proportional to $\omega$. The decay parameter $\alpha$ is related to the friction stress that appears through mutual annihilation. So, as a first approximation, we can put $\alpha=\alpha_0\omega$. ($\alpha_0$ is a constant.) Therefore, the parameter $l$ is led to

$$l(\sigma) = \frac{1}{2\pi\alpha_0}\exp\left(-\frac{Q}{\sigma}\right), \qquad (2\text{-}8)$$

which does not depend on $\omega$. Consequently, from equations (2-7) and (2-8), the general feature that S-N curves hardly depend on $\omega$ is derived in this case. We have to note that our results for S-N curves might depends on $\omega$ through $\sigma$ and the constant $\alpha=\alpha_0\omega+\alpha_1\omega^2+\cdots$.

Except for the neighborhood of the endurance limit, however, the parameter $\alpha$ can be taken to be zero, i.e. $l=\infty$. In this case, mean life $\overline{N}_{ci}$ is expressed as

$$\overline{N}_{ci} = (1 + \frac{U}{\sigma})\exp(\frac{Q}{\sigma}). \qquad (2\text{-}9)$$

The derivation of equation (2-9) is given in Appendix 2-2.

Before we calculate the mean life from the equation (2-9), we investigate the expression for $\sigma$, i.e. the average value of the random variable $Y$ or $X$. In our model $\sigma$ is a value per one dislocation source.

Now, when cyclic stress $S\sin(\omega t)$ is loaded on materials the cyclic strain $\varepsilon\sin(\omega t-\phi)$ is induced where $\varepsilon$ and $\phi$ denote strain amplitude and the phase lag due to the internal friction, respectively. Consequently, it is natural to assume that the energy which is stored transiently inside the materials per unit cycle and unit volume is proportional to

$$\oint S\sin\omega t \times d\{\varepsilon\sin(\omega t + \phi)\}.$$

So, it seems reasonable that the average value of the energy per unit volume, $\overline{Z}$, is proportional to $S^2 \sin\phi$. We put

$$\overline{Z} = c_0 S^2/E,$$

where $E$ stands for elastic modulus and $c_0$ is a constant. Here, we assume that the energy $\overline{Z}$ is shared equivalently among the active dislocation multiplication sources. Therefore, if the number density of the active dislocation multiplication sources per unit volume is denoted by $n(S)$ (it must be considered that $n(S)$ depends on $S$), we put

$$\sigma = \overline{Z}/n(S),$$

i.e.

$$\sigma = \frac{c_0 S^2}{E} \cdot \frac{1}{n(S)}. \qquad (2\text{-}10)$$

Accordingly, if the expression for $n(S)$ which is derived in the next section is used, we can obtain the average life up to cell-sized crack initiation, substituting equation (2-10) into equations (2-9) or (2-6).

Using equations (2-9) and (2-10), we consider the growing process of cell-sized crack in order to obtain the life up to grain-sized crack initiation. As the appropriate treatment for the propagation of such cell-sized cracks is unknown, we assume as follows. When a cell-sized crack has crossed a cell near the material's surface, the fatigue damage begins to accumulate in the adjacent cell and some further cycles are spent before the crack can cross the adjacent cell, i.e. the accumulated energy in the cell (strain energy) exceeds $U$. Then, the crack propagates across the cell. This process is repeated until the crack depth attains a specific value $a$. If we assume that the life of each cell from the begining of damage accumulation up to failure is a random variable with the same distribution and mutually independent, the life up to the initiation of a crack with a depth of $a$ is the summation over the failing life of each cell where the crack propagated. Since the average value of each cell's life is mutually equivalent $\overline{N}_a$, the mean life up to the initiation of a crack with a depth of $a$, is equal to the product of $\overline{N}_{ci}$ and $a/d(S)$, where $d(S)$ denotes the size of cell, i.e.

$$\overline{N}_a = \overline{N}_{ci} \times \frac{a}{d(S)} . \tag{2-11}$$

As the curves of crack initiation in experiments are those of the grain-sized crack initiation, $a$ is taken to be the depth of the crack whose surface length is equal to grain size $d_g$. The relation between the surface length of crack $\xi$ and its depth $\eta$ is obtained experimentally, i.e. the following relation is reported[34],

$$\eta = \gamma\xi,$$

where $\gamma$ is a constant. We assume that, for the crack whose length is less than $d_g$, this relation also holds, so the depth of a crack whose surface length is $d_g$ is taken to be $\gamma d_g$. Consequently, the mean life up to grain-sized crack initiation, $\overline{N}_{gi}$ is given by

$$\overline{N}_{gi} = \frac{\gamma d_g}{d(S)} \overline{N}_{ci}, \tag{2-12}$$

from equation (2-11). Equations (2-9) and (2-12) yield

$$\overline{N}_{gi} = \frac{\gamma d_g}{d(S)}(1 + \frac{U}{\sigma})\exp(\frac{Q}{\sigma}), \tag{2-13}$$

where

$$\sigma = \frac{c_0 S^2}{E} \cdot \frac{1}{n(S)} .$$

$d(S)$ and $n(S)$ are investigated in the next section.

## 2-3 The Size of Cells

In this section, making use of a simple model, we will specify expressions for $n(S)$ and $d(S)$.

Under the repeated stress, dislocations in virgine materials are redistributed. Some parts of the dislocations act as dislocation sources, and produce new dislocations. Subsequently, they yield microscopic structures, named cell. Consequently, we assume that the sizes of cells are determined by the distribution and density of dislocation-segments in the early stage. The motion of a dislocation is impeded by point defect clusters or other dislocations, so that the dislocation is divided into dislocation-segments. Since the pinning points of dislocations distribute randomly and are mutually independent, we regard the length of a dislocation segment $L$ as a random variable with exponential distribution whose mean is $l_0$. Here, we note the followings; Schlipf et al.[35] showed that the distribution of the length of dislocation-segments for the equiliblium state can be approximated by an exponential distribution.

From the equiliblium condition between stress and line tension of the dislocation, it is natural to consider that only the dislocation-segments longer than

$$l_c = \delta\mu b/S, \tag{2-14}$$

have possibilities of dislocation multiplication under the stress $S$, where $\mu$ is the rigidity modulus, $\delta$ a numerical constant of order one and $b$ the absolute value of Burgers vector. Hence the probability that a specified dislocation-segment can act as a dislocation multiplication source, is

$$\kappa \mathrm{Pr.}\{L > l_c\} = \kappa \exp(-l_c/l_0), \tag{2-15}$$

where $\kappa$ is the probability that the dislocation-segment has an adequate configuration to glide under the stress. If $d_0$ denotes the dislocation density in the early stage, $n(S)$ is obtained from equations (2-14) and (2-15) as follows;

$$n(S) = \frac{\kappa d_0}{l_0}\exp(-\frac{l_c}{l_0}) = \frac{\kappa d_0}{l_0}\exp(-\frac{\delta\mu b}{l_0 S}), \tag{2-16}$$

since $d_0/l_0$ is the average number of dislocation segments per unit volume. Therefore the average mutual distance between two active dislocation sources is given by

$$\left(\frac{1}{n(S)}\right)^{\frac{1}{3}} = \left(\frac{l_0}{\kappa d_0}\right)^{\frac{1}{3}} \exp\left(\frac{\delta\mu b}{3 l_0 S}\right) . \tag{2-17}$$

The size of microscopic structure formed under the repeated stress corresponds to the mutual distance of two active dislocation sources, so that the average size of cells is proportional to equation (2-17), i.e.

$$d(S) = \beta\left(\frac{1}{n(S)}\right)^{\frac{1}{3}} = \beta\left(\frac{l_0}{\kappa d_0}\right)^{\frac{1}{3}} \exp\left(\frac{\delta\mu b}{3 l_0 S}\right) , \tag{2-18}$$

where $\beta$ is a constant. We compare our results, equation (2-18), with the experimental data of the size of cell formed under the repeated stress. The logarithms of the average cell diameters measured for copper[36] and 0.07%C Fe[37] are plotted for inverse stress amplitude in Figs. 2-1 and 2-2, respectively. From these figures, the experimental results seem to lie on a line. Since equation (2-18) is expressed as a line in Figs. 2-1 and 2-2, we see that the relation (2-18) explains the $S$-dependence of $d(S)$. (The parameter value of solid lines in Figs. 2-1 and 2-2 are $l_0=0.08\mu$m, $d_0=8.1\times10^{11}$m$^{-2}$ and $l_0=0.018\mu$m, $d_0=2.0\times10^{13}$m$^{-2}$, respectively, if we put $\beta=\delta=\kappa=1$.)

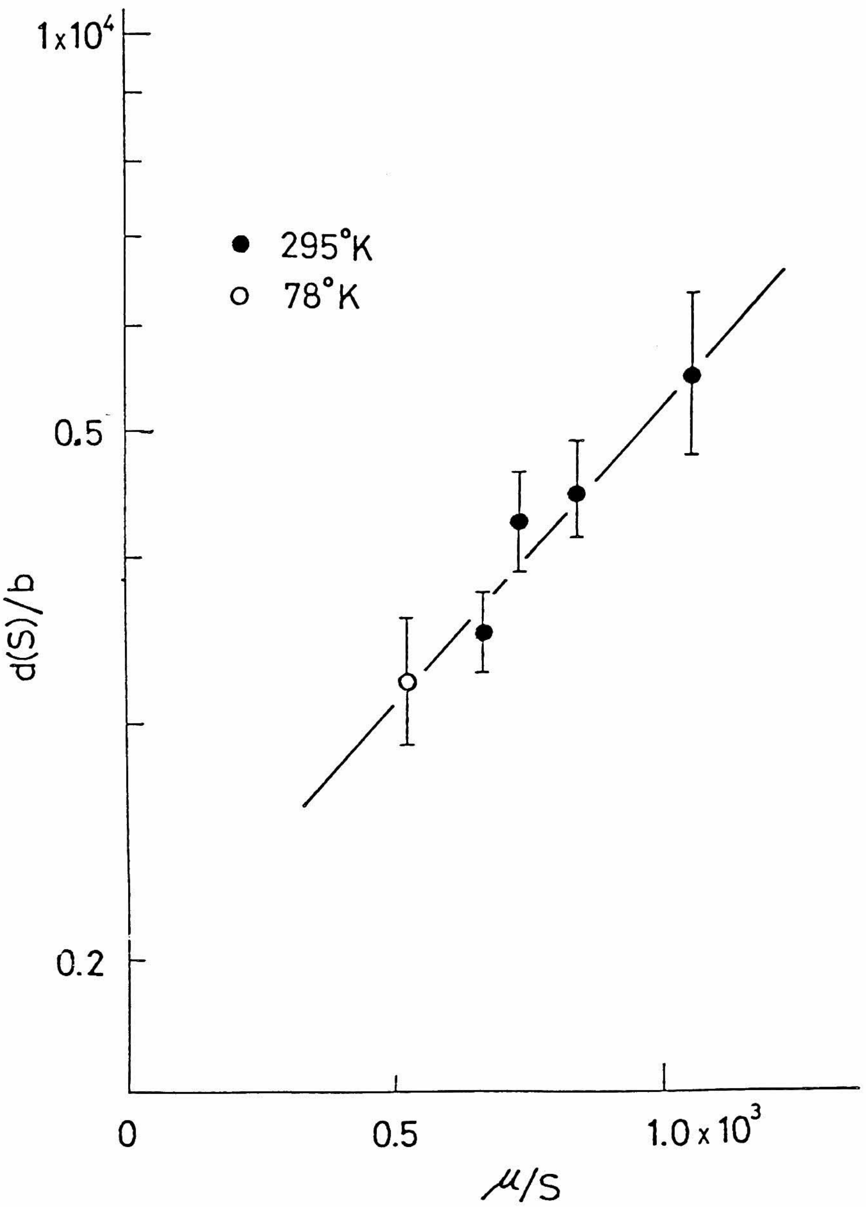


Fig. 2-1 Cell size *d(S)* of copper formed under repeated stress[36].

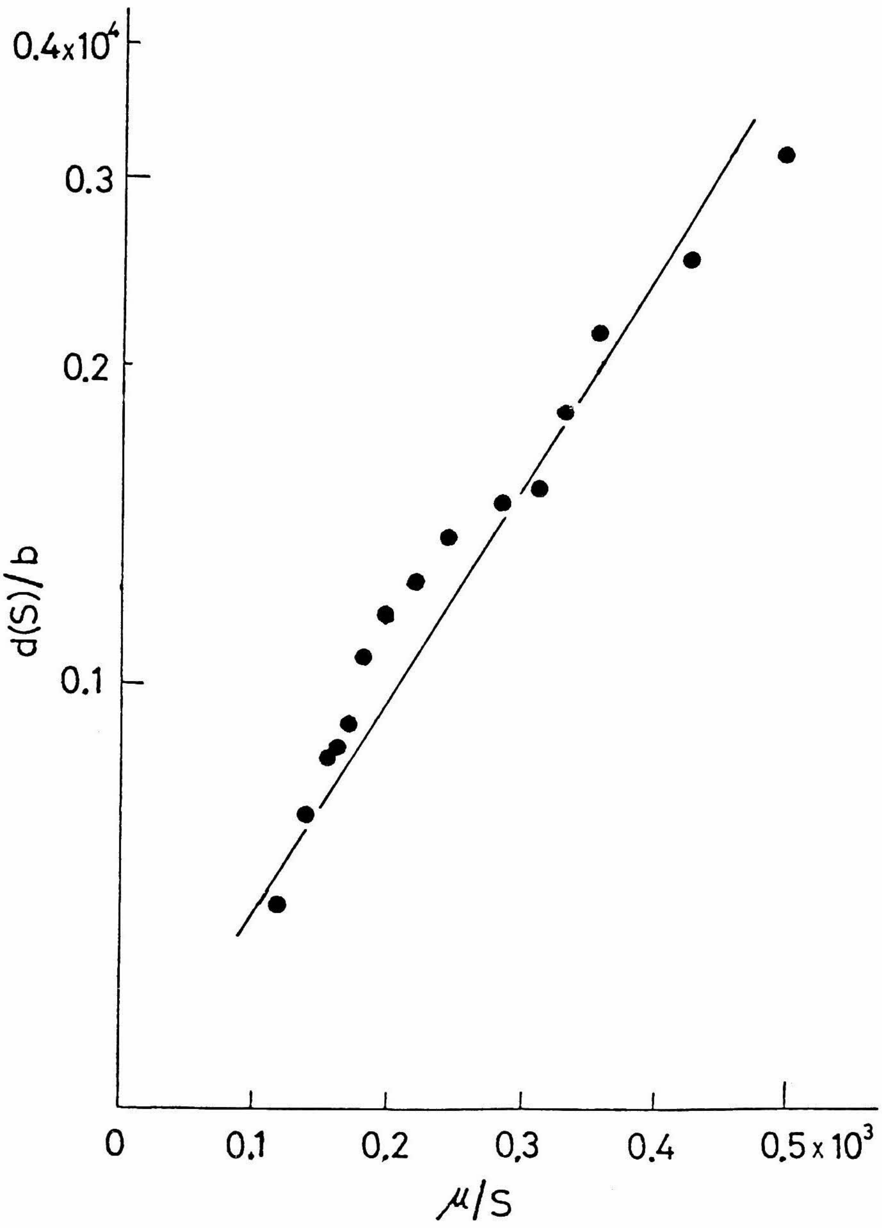


Fig. 2-2 Cell size *d(S)* of 0.07%C Fe formed under repeated stress[37].

## 2-4 Estimation of Some Parameters

In order to estimate the life up to crack initiation from the forgoing model, we need to know the values of the parameters. Among them we focus our attention on the value of $Q/U$ which has close relations with the static material properties[38],[39], i.e. we give an approximate method for determining the value of $Q/U$. Since $Q_0$ and $U_0$, the values per unit volume of $Q$ and $U$, can be related to the static stress-strain curve, we can determine $Q_0$ and $U_0$. Moreover, we assume that the value of $Q/U$ is approximated by the value of $Q_0/U_0$, i.e.

$$\frac{Q}{U} = \frac{Q_0}{U_0} .$$

The energy $Q_0+U_0$ is assumed to correspond to the area under the static stress-strain curve before necking [39]. That is to say, if $S(\varepsilon)$, $\varepsilon_n$ and $\varepsilon_l$ denote the static true stress-strain curve, the true strain when the necking just occurs and the true strain corresponding to the end of Lüders strain, respectively, we can put

$$Q_0 + U_0 = \int_{\varepsilon_l}^{\varepsilon_n} S(\varepsilon) d\varepsilon. \tag{2-19}$$

Here, we exclude the energy necessary for formation of the Lüders bands from $Q_0+U_0$ because we are interested in the saturation hardening stage.

As stated in the previous section, the part of the transiently stored energy less than $Q_0$ makes no direct contribution to fatigue damage. It can be considered that this part is mainly expended on the reversible flip-flop motion of dislocations, i.e. the anelastic energy. If an energy larger than $Q_0$ is supplied, however, the dislocations bow out beyond the critical radius and dislocation multiplication takes place. In the saturation hardening stage, the value of $Q_0$ must be such that new dislocations are seldomly generated due to point defect clusters or dislocations which impede their formation. If we divide $Q_0+U_0$ into the anelastic part $Q_0$ and the plastic part $U_0$ at the sudden change point of $S(\varepsilon)$, the above condition can be satisfied. Accordingly the value of $Q_0/U_0$, namely, $Q/U$ is obtained from the static properties of materials.

## 2-5 A Comparison of Analytic and Experimental Results

From the argument in the previous section, the average life up to grain-sized crack initiation, $\overline{N}_{gi}$, is obtained by equations (2-13), (2-16) and (2-18).

$$\overline{N}_{gi} = \frac{\gamma}{\beta} d_g \left(\frac{\kappa d_0}{l_0}\right)^{\frac{1}{3}} \exp\left(-\frac{\delta \mu b}{3 l_0 S}\right)$$
$$\times \exp\left(\frac{Q}{\frac{c_0 S^2}{E} \frac{l_0}{\kappa d_0} \exp\left(\frac{\delta \mu b}{l_0 S}\right)}\right) \times \left(1 + \frac{U}{\frac{c_0 S^2}{E} \frac{l_0}{\kappa d_0} \exp\left(\frac{\delta \mu b}{l_0 S}\right)}\right) . \qquad (2\text{-}20)$$

The determination of the values of the parameters $Q/U$, $\gamma$, $\beta$, $\delta/l_0$, $Q/c_0$ and $\kappa d_0/\delta$ in equation (2-20) are made as follows. First of all $Q/U$ is determined from the static stress-strain relation as stated in section 2-4. The value of $\gamma$ which is the ratio of depth of a crack to its surface length is reported by some workers. Here, we put $\gamma=0.38$, the experimental value reported by Nakazawa et al.[34]. Furthermore, if we put $\beta=1$, the values of three parameters, i.e. $\delta/l_0$, $Q/c_0$ and $\kappa d_0/\delta$, are left undetermined.

Our results are compared with the experiments using three kinds of low-carbon steel S20C with different grain sizes[31]. Hereafter, the materials with grain size

$d_g$=7.8, 20.5 and 55μm are designated as A, B and C, respectively. Their chemical composition and mechanical properties are shown in Tables 2-1 and 2-2.

Now, we must determine the values of $\delta/l_0$, $Q/c_0$ and $\kappa d_0/\delta$ for each material. In the first place, we do it for the material C. In our numerical calculation, the inclination of the S-$\overline{N}_{gi}$ curve is roughly independent on the values of $Q/c_0$ and $\kappa d_0/\delta$, and practically determined if the value of $\delta/l_0$ is specified. So, we choose the value of $\delta/l_0$ with which the inclination of the S-$\overline{N}_{gi}$ curve coincides with the experimental results. We put $\kappa d_0/\delta=10^5 \mathrm{mm}^{-2}$ tentatively, so we can chose the value of $Q/c_0$ in order that the S-$\overline{N}_{gi}$ curve is in agreement with the experimental results. For $Q/c_0(\kappa d_0/l_0)=4.18$ MPa, the S-$\overline{N}_{gi}$ and S-$\overline{N}_{ci}$ curves are shown in Fig. 2-3. Our results are in agreement with the experimental results. In Fig. 2-3, fatigue lives up to cell-sized crack initiation are slightly greater than the experimental data for the lives up to the formation of slip bands, and their inclinations are roughly equal.
Here, we show the adequacy of $\kappa d_0/\delta=10^5 \mathrm{mm}^{-2}$. If we choose the value of $\kappa d_0/\delta$ greater than $10^5 \mathrm{mm}^{-2}$, the average life up to cell-sized crack initiation becomes smaller than the life up to the formation of slip bands. It is inconsistent.

Table 2-1 Chemical composition of material used (%) [32].

| C | Mn | Si | P | S | N |
|---|---|---|---|---|---|
| 0.20 | 0.92 | 0.26 | 0.11 | 0.15 | 0.009 |

Table 2-2 Heat-treatment condition and mechanical properties of materials[32].

| Materials | Heat-treatment post machining | Grain size $d_g$ [μm] | fatigue limit $S_w$ [MPa] | Yield strength $S_Y$ [MPa] | Tensile strength $S_B$ [MPa] |
|---|---|---|---|---|---|
| A | Annealed at 900°C for 10 min followed by air cooling | 7.8 | 235 | 366 | 528 |
| B | Annealed at 1000°C for 1 hr followed by furnace cooling | 20.5 | 178 | 275 | 466 |
| C | Annealed at 1200°C for 5 hr followed by furnace cooling | 55.0 | 163 | 194 | 433 |

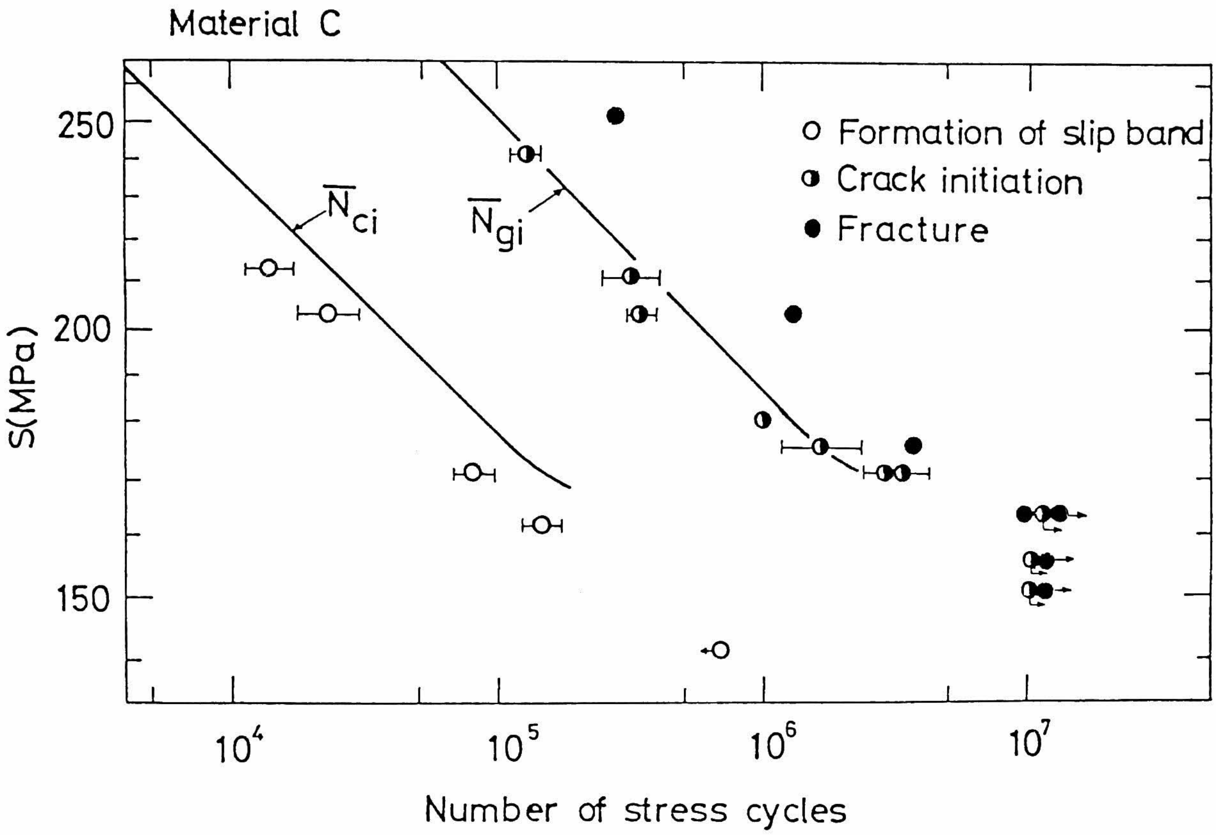


Fig. 2-3 Fatigue lives up to cell- and grain-sized crack initiation for material C.

Subsequently, we investigate the materials A and B. It may be considered that the values of $\delta/l_0$, $\kappa l_0/\delta$ and $Q/c_0$ are mutually different for each material. It is assumed, however, that $\delta/l_0$ and $\kappa d_0/\delta$ take the same values for materials A, B and C. As a result, the value of $Q/c_0$ for materials A and B is left undetermined. As stated above, $Q/c_0$ is the parameter by which the position of the S-$\overline{N}_{gi}$ curve is determined, so we can specify the value of $Q/c_0$ from the comparison with the experimental data. In table 2-3, we show the chosen values of the parameters. The S-$\overline{N}_{ci}$ and S-$\overline{N}_{gi}$ curves calculated using these parameters are shown in Figs.2-4 and 2-5 for materials A and B, respectively. From these figures, we see our theory well reproduces the average life up to grain-sized crack initiation which is obtained by Taira et al.[31]. The smaller $d_g$ becomes, the greater the difference between the life up to the formation of slip bands and $\overline{N}_{ci}$ becomes, but the inclination of the S-$\overline{N}_{ci}$ curve seems to have the same value as one of the slip formation curves. Besides, if we put $\kappa=1$ and $\delta=1$, we obtain $d_0=10^5\mathrm{mm}^{-2}$ and $l_0=0.1\mu\mathrm{m}$. It can be considered that these values are adequate. Following the line of the foregoing argument, we can conclude that the fatigue crack initiation can be well described by the use of our model proposed in this chapter.

Table 2-3 Values of parameters in equation (2-20) for each material.

| Materials | $\frac{Q}{U}$ | $\frac{Q}{c_0}\left(\frac{\kappa d_0}{l_0}\right)$ [MPa] | $\frac{\kappa d_0}{\delta}$ [$m^{-2}$] | $\gamma$ | $\frac{l_0}{\delta}$ [μm] |
|---|---|---|---|---|---|
| A($d_g$=7.8μm) | 0.53 | 6.71 | 1.0×10$^{11}$ | 0.38 | 0.099 |
| B($d_g$=20.5μm) | 0.32 | 4.95 | | | |
| C($d_g$=55.0μm) | 0.48 | 4.18 | | | |

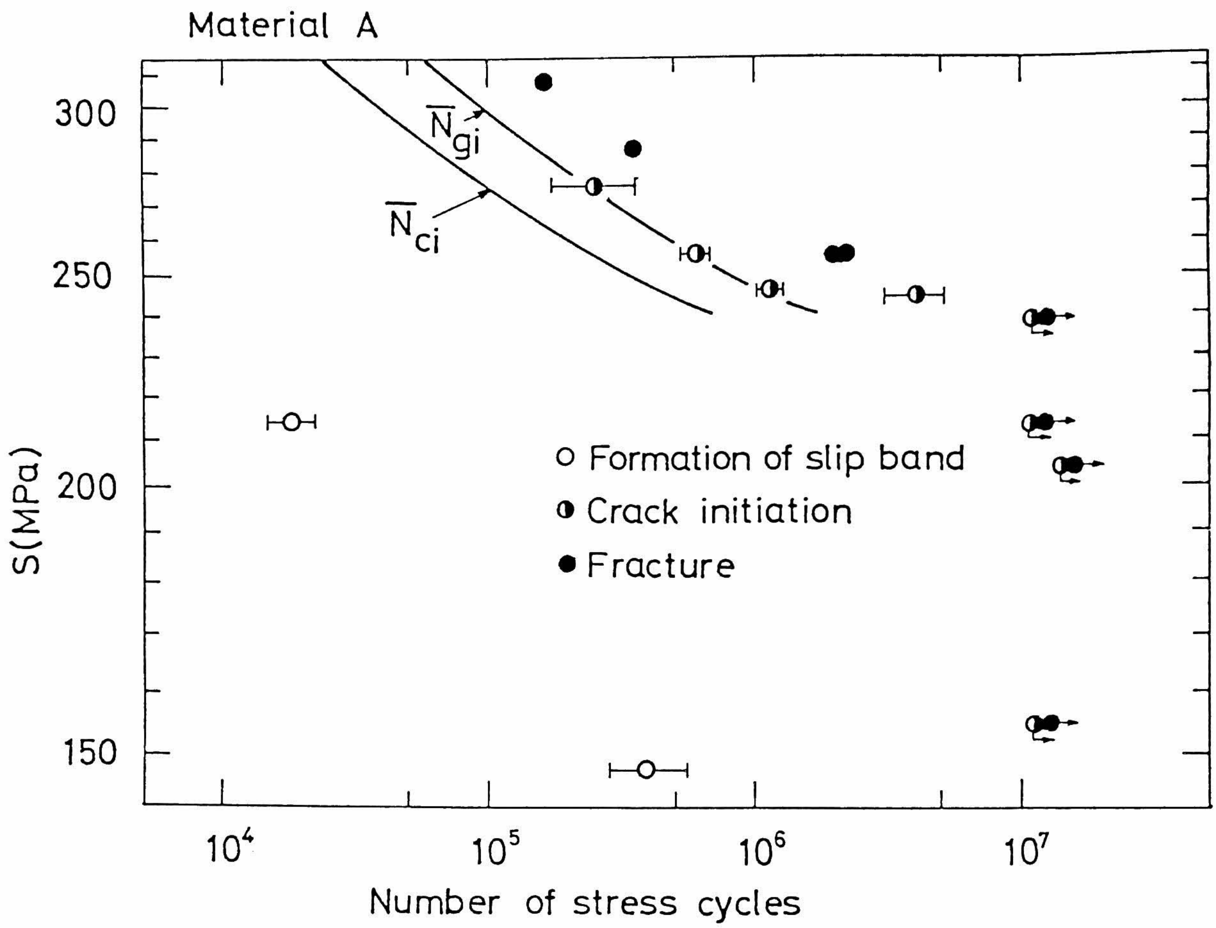


Fig. 2-4 Fatigue lives up to cell- and grain-sized crack initiation for material A.

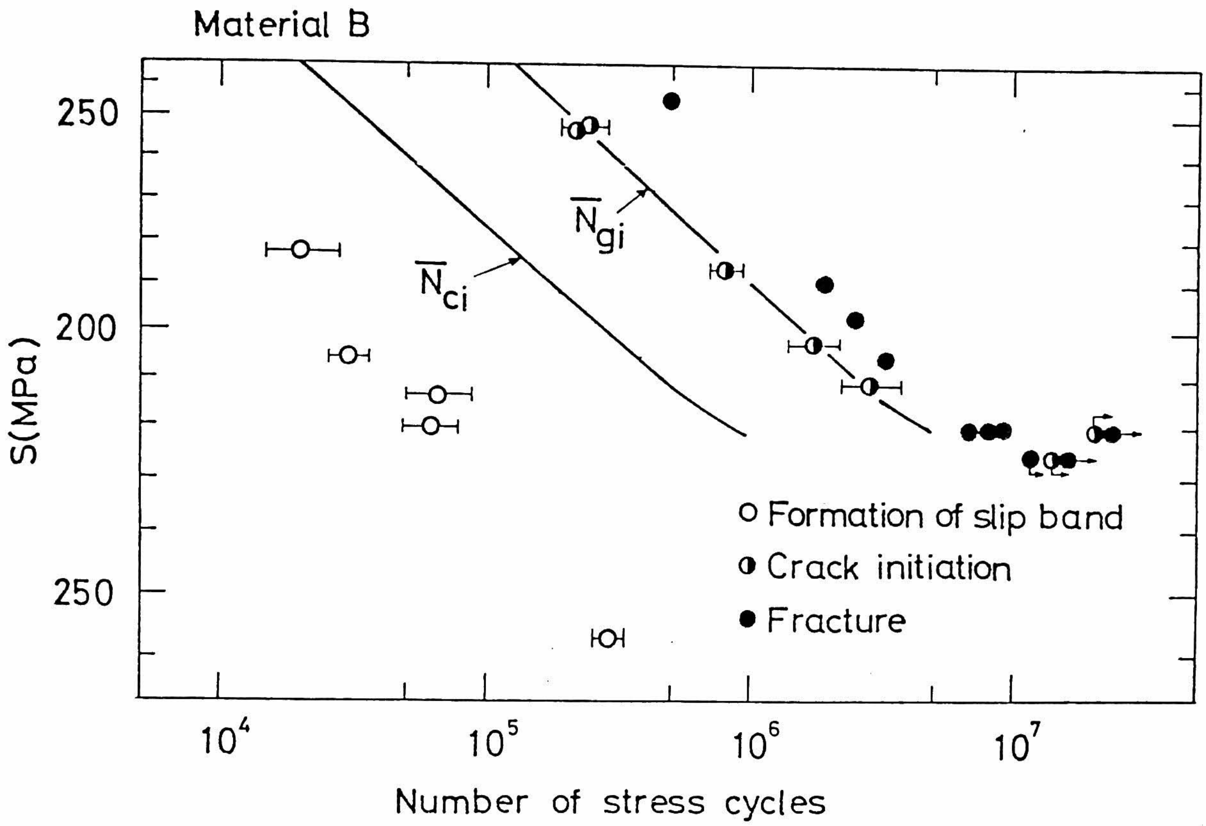


Fig. 2-5 Fatigue lives up to cell- and grain-sized crack initiation for material B.

## 2-6 Some Remarks

In this chapter we have proposed a damage accumulation model, and investigated the fatigue crack initiation. With the use of this model, we have revealed the following.

(1) Under cyclic stress, fatigue damage accumulates and a crack appears. The size of the crack is of the order of the scale of microscopic structures due to piling-up of dislocations.

(2) We can obtain the average life up to grain-sized crack initiation under the assumption for the growing process of cell-sized cracks, that life from the begining of damage accumulation up to the initiation of a crack in each cell is mutually independent and has the same distribution.

(3) From a probabilistic argument, we can obtain the stress dependence of cell size.

(4) The experimental results for three kinds of low-carbon steel with the different grain sizes are well explained in the model proposed in this chapter.

## Appendix 2-1 The Derivation of Equation (2-3)

We give a brief outline of the derivation of equation (2-3). Further detailed argument is shown in reference [40].

Let $S_n$ $(n=1,2,\cdots)$ denotes a random variable that expresses the moment of occurrence of the n'th event. $\{N(t),\ t>0\}$, the number of occurrence of the event in time $t$, is a Poisson process with parameter $\lambda$ and $X_n$ $(n=1,2,\cdots)$ a random variable with distribution $H(x)$ which expresses the magnitude of damage generated by the n'th event. Consider a cumulative process

$$Z(t) = \begin{cases} \sum\limits_{n=1}^{N(t)} X_n \exp\{-\alpha(t - S_n)\} & \text{for } N(t) > 0 \\ 0 & \text{, otherwise} \end{cases} \qquad (2\text{-}21)$$

which denotes the total amount of damage at time $t$. What we seek is the mean value of time $T$ that $Z(t)$ exceeds a pre-specified value $U$ for the first time.

Let us introduce the probability $f(z,t)dz$ that $Z(t)$ exceeds $U$ for the first time and falls into an interval $(z,z+dz)$ $(z>U)$ at time $t$, given that it has never exceeded $U$ up to time $t$. Then it can be shown that the function

$f(z,t)$ satisfies the integral equation

$$f(z,t) = -\int_0^U \lambda\gamma(x,t\,|\,0)\,dH(z - x)$$

$$+ \lambda\int_0^t ds\int_U^\infty dy\int_0^U \gamma(x,s\,|\,Y)f(y,t-s)\,dH(z-x), \tag{2-22}$$

where

$$\gamma(x,t\,|\,y)\,dxdy$$

$$= \Pr.\{x < z(s+t) \leq x+dx\,|\,y < z(s) \leq y+dy\}. \tag{2-23}$$

For our case the explicit expression of $\gamma(x,t|y)$ can be found, and then by the use of Laplace transforms the formal solution of equation (2-22) can be obtained.
If $\lambda/\alpha = l$ is an integer, after some cumbersome calculations we obtain the Laplace transform of $f(z,t)$ defined by

$$\hat{f}(z,p) = \int_0^\infty \exp(-pt)f(z,t)\,dt$$

as follows:

$$\hat{f}(z,p)=\frac{\exp(-\frac{z}{\sigma})\frac{1}{p+1}\sum_{n=0}^{l}\frac{(l-n+1)_n(\frac{U}{\sigma})^n}{(pl+l-n)_n\ n!}}{1+\frac{U}{\sigma}\exp(-\frac{U}{\sigma})\sum_{m=0}^{\infty}\sum_{n=0}^{l-1}\frac{(l-n)_{n+1}(n+2)_n(\frac{U}{\sigma})^{m+n}}{(pl+l+m-n-1)_{2n+2}\ m!n!}},$$

(2-24)

where

$$(\beta)_n = \beta(\beta+1)\cdot\cdots\cdot(\beta+n-1),$$

and

$$(\beta)_0 = 1$$

Since $\int_U^\infty \hat{f}(z,p)dz$ is the Laplace transform of the probability density function of the random variable $T$, it follows from equation (2-21) that

$$\lambda\overline{T} = F_l(\frac{U}{\sigma}) = -\lim_{p\to 0}\frac{d}{dp}\int_U^\infty \hat{f}(z,p)dz$$

$$
= \frac{l!}{(U/\sigma)^{l}} \Bigg\{ \exp\left(\frac{U}{\sigma}\right) + \frac{U}{\sigma} \sum_{n=0}^{l-2} \sum_{m=0}^{\infty} \frac{(l-n)_{n+1}(n+2)_{n}(U/\sigma)^{m+n}}{(l+m-n-1)_{2n+2}\, m!\, n!}
$$

$$
+ \sum_{m=1}^{\infty} \frac{(l)_{l}\left(\frac{U}{\sigma}\right)^{m+l}}{(m)_{2l}\, m!} - \sum_{m=0}^{l-1} \frac{l\left(\frac{U}{\sigma}\right)^{m}}{(l-m)\, m!} \Bigg\} \qquad (2\text{-}25)
$$

$$
+ l \sum_{m=l+1}^{2l-1} \frac{1}{m} .
$$

## Appendix 2-2 The Derivation of Equation (2-9)

We can derive equation (2-9) from equation (2-6) if we put $l=\infty$, but we can obtain it directly as follows.

The probability density function of the life measured up to the time of the cell-sized crack initiation is denoted by $f(t)$. Since $f(t)dt$ is the probability that between $t$ and $t+dt$ the accumulated damage (stored energy) exceeds $U$ for the first time, the following relation holds,

$$f(t)dt = \int_0^U dx g(x,t)\lambda\exp\{-(U-x)/\sigma\}dt, \qquad (2\text{-}26)$$

where $g(x,t)dx$ is the probability that the strain energy is between $x$ and $x+dx$ at time $t$, and $\lambda\exp\{-(U-x)/\sigma\}dt$ the probability that the event $\mathcal{A}$ occurs between $t$ and $t+dt$ and the energy larger than $U-x$ is accumulated at $t$. Since the event $\mathcal{A}$ occurs according to a Poisson process with parameter $\lambda$, the probability that the event occurs $n$ times in time interval $t$ is

$$\frac{(\lambda t)^n}{n!}\exp(-\lambda t). \quad (n=0,1,2,\cdots). \qquad (2\text{-}27)$$

With the use of equation (2-27), $g(x,t)$ is given by

$$g(x,t)=\delta(x)\exp(-\lambda t)+\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\exp(-\lambda t)g_n(x), \tag{2-28}$$

where $\delta(x)$ is Dirac's delta function, and $\lambda$ given in equation (2-2). $g_n(x)$ is the probability density function for the accumulated energy, given that the event $\mathcal{A}$ occurs $n$ times. The accumulated energy for the case that the event $\mathcal{A}$ occurs once is the exponential distribution whose mean is $\sigma$, so $g_n(x)$ is expressed as

$$g_n(x) = \frac{x^{n-1}}{\sigma^n(n-1)!}\exp(-x/\sigma). \tag{2-29}$$

Therefore, substituting equations (2-28) and (2-29) into equation (2-26), we can finally obtain the following.

$$f(t) = \lambda\exp(-\lambda t-\frac{U}{\sigma})\, I_0(2\sqrt{\frac{U}{\sigma}\lambda t}), \tag{2-30}$$

where $I_0(x)$ is the modified Bessel function of order zero, its explicit form being

$$\mathrm{I}_0(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{2n}}{(n!)^2} . \tag{2-31}$$

From equation (2-30), the average time $\overline{T}$ up to cell-sized crack initiation is

$$\overline{T} = \int_0^{\infty} dt \cdot t f(t) = \frac{1}{\lambda}(1 + \frac{U}{\sigma}). \tag{2-32}$$

So, from equation (2-32) the average life $\overline{N}_{ci}$ (measured by cycle number) is obtained as follows.

$$\overline{N}_{ci} = \frac{\omega}{2\pi}\overline{T} = (1+\frac{U}{\sigma})\exp(\frac{Q}{\sigma}). \tag{2-33}$$

# CHAPTER 3

# A PROPAGATION LAW OF SMALL FATIGUE CRACKS

## 3-1 Introduction

The greater part of fatigue life for a smooth specimen is spent on initiation of a crack and growth of the small crack. In Chapter 2 we proposed a stochastic accumulation model for fatigue damage which is based on a dislocation multiplication model, and obtained an expression for fatigue life up to grain-sized crack initiation. In this chapter, we investigate a propagation law of small fatigue cracks(one grain size order).

It is well established that the propagation process of through cracks under repeated stress can be treated on the basis of linear fracture mechanics. On the contrary, however, many workers reported that the growth law of small cracks is quite different from one of through cracks and especially that in some materials the growth rate of small cracks cannot be treated as a function of stress intensity range[18],[19],[20]. We investigate the small crack growth, that is, crack growth in Stage $II_a$, from the same assumptions as in the previous chapter.

It is reported that the propagation rate of creep cracks is proportional to the modified J-integral $J'$ (creep J-integral)[41]. Ihara focussed his attention on the creep crack propagation and showed that creep crack propagation rate is proportional to creep J-integral calculated with an expression obtained on the basis of a dislocation multiplication model[42]. We obtain an expression for growth rate of a small fatigue crack by the analogy of the creep crack growth law, and compare the results with experiments. Furthermore, life up to fatigue failure is estimated with the use of this growth law.

In section 3-2. a propagation law of small fatigue cracks is explained. In section 3-3, our results are compared with the experimental data. Some remarks are included in section 3-4. In Appendix 3-1, the definition and physical meaning of the J-integral are given.

## 3-2 A Propagation Law of Small Fatigue Cracks

In a recent paper, Ihara has investigated growth of creep cracks[42]. Since his treatment is closely related to our treatment for the growth of small cracks, it is described in detail.

We consider a material in which strain $\varepsilon$ is proportional to some power of tension stress $S$,

$$\frac{\varepsilon}{\varepsilon_0} = \alpha\left(\frac{S}{S_0}\right)^m, \tag{3-1}$$

where $\varepsilon_0$ and $S_0$ are the reference values for stress and strain, $\alpha$ is a constant and $m$ is the hardening exponent. In this case J-integral of the material with a shallow crack of depth $a$ is expressed by

$$J = \varepsilon S a f(m), \tag{3-2}$$

where $f(m)$ is a definite function of the exponent $m$[43]. The detailed explanation of this expression is given in Appendix 3-1.

Under a uniform stress which is so large that creep phenomena is remarkable, microcracks are formed in materials.

In some materials creep failure occurs by the initiation and propagation of a single macroscopic crack. Fracture mechanics provides the means by which such creep behavior is analyzed. The growing phenomenon of fatigue through cracks can be successfully treated by linear fracture mechanics, but the high-temperature creep phenomenon is not a typical linear elastic phenomena. Therefore, the growth rate of creep cracks may not be regarded as a function of the linear elastic parameter (stress intensity factor $K$). Hence, a different parameter $J'$ which is a modified J-integral is proposed, and named creep J-integral. $J'$ is simply a modification of the J-integral (equation (3-24)) where strain and displacement $\varepsilon_{ij}$ and $u_i$ are replaced by their rate $\dot{\varepsilon}_{ij}$ and $\dot{u}_i$. Consider a homogeneous body which contains a crack having a flat surface Suppose the body is subjected to a two-dimensional deformation field $S_{ij}$. The y-axis is taken perpendicular to the crack surface and the x-axis is taken perpendicular to the y-axis and the crack edge. $S_{ij}$ depends only on $x$ and $y$. If strain rate $\dot{\varepsilon}$, which is $[\dot{\varepsilon}_{ij}]$, the infinitesimal strain rate tensor, has a definite value, $J'$ can be defined by

$$J' = \int_{\Gamma} \{W' dy - T_i (\frac{\partial \dot{u}_i}{\partial x}) dl\} , \qquad (3\text{-}3)$$

where $W'$ is the strain energy rate density associated with $S_{ij}$ and $\dot{\varepsilon}_{ij}$, i.e.

$$W' = \int_0^{\dot{\varepsilon}} S_{ij} d\dot{\varepsilon}_{ij} \ , \qquad (3\text{-}4)$$

and $T_i$ is tension vector defined by

$$T_i = S_{ij} n_j \ ,$$

where $n_j$ is the outward normal along $\Gamma$. $\Gamma$ is the line contour in a plane parallel to the x-y-plane, and surrounds the crack tip. The integral is evaluated in a counter-clockwise sense from the lower flat crack surface to the upper flat surface. $l$ is the arc length along $\Gamma$.
It can be shown that $J'$ is independent of the integral path as in the J-integral. If a creep law is given by

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} = \alpha' \left(\frac{S}{S_0}\right)^{m'} , \qquad (3\text{-}5)$$

where $\dot{\varepsilon}_0$ and $S_0$ are the reference values for strain rate and stress, respectively, and $m'$ and $\alpha'$ are constants, $J'$ is given by

$$J' = \dot{\varepsilon} S a f(m') \text{ [44]}. \tag{3-6}$$

This formation can be derived in the same way as equation (3-2). It is reported that creep crack growth rate for through crack $da/dt$ is proportional to the modified J-integral $J'$ [41] which is commonly evaluated in experiments by the use of the following approximate expression,

$$J' = g(m')\dot{V}_0 S_{net} , \tag{3-7}$$

where $\dot{V}_0$ is the crack opening displacement rate, $S_{net}$ is net stress, and $g(m)$ is a definite function of $m$. That is to say, $da/dt$ is given by

$$\frac{da}{dt} = c_1 g(m') \dot{V}_0 S_{net} , \tag{3-8}$$

where $c_1$ is a constant. Furthermore, for a crack with a small depth, the following relation holds[41],

$$\dot{V}_0 \propto a\dot{\varepsilon}. \tag{3-9}$$

From the foregoing two facts, i.e. equations (3-8) and (3-9), it can be considered that *da/dt* for small cracks is expressed by

$$\frac{da}{dt} = c_1 J' = c_2 \dot{\varepsilon} S a, \tag{3-10}$$

where $c_1$ and $c_2$ are constants, since in this case $S_{net}$ can be replaced by $S$ (applied stress).

The steady-state creep rate $\dot{\varepsilon}_s$ obtained with the use of dislocation multiplication model[45] behaves as if it were proportional to some power of stress amplitude, i.e. it can be expressed by equation (3-5). So, we can calculate $J'$, substituting the calculated $\dot{\varepsilon}_s$ into $\dot{\varepsilon}$ in equation (3-6). Doing so, we obtain a proportional law of small creep cracks from equation (3-10). On the basis of this law, life of creep crack propagation is obtained and the result is in agreement with the experimental data[42].

In low-cycle fatigue, the following relation is observed by Shiratori et al.[46],

$$\frac{d\varepsilon}{dn} = \alpha'' S^{m''} , \tag{3-11}$$

where $d\varepsilon/dn$ is an increase of maximum strain per cycle, $S$ is stress amplitude, and $\alpha''$ and $m''$ are constants. Equation (3-11) corresponds to equation (3-5) in creep phenomena. In high-cycle fatigue, it can be considered that equation (3-11) holds in the plastic region near the crack tip. We can assume that the size of plastic zone near the crack tip is much larger than the size of crack for Stage $II_a$ crack. So, on the analogy of creep crack growth, modified J-integral $J^{(N)}$ ( fatigue J-integral) can be defined in correspondence to $J'$ (creep J-integral), and can be expressed by

$$J^{(N)} = \frac{d\varepsilon}{dn} S a f(m''). \tag{3-12}$$

Furthermore, the propagation rate of small fatigue cracks $da/dn$ is assumed to be proportional to $J^{(N)}$, i.e.

$$\frac{da}{dn} = c_3 J^{(N)} = c_4 \frac{d\varepsilon}{dn} S a \;, \tag{3-13}$$

where $c_3$ and $c_4$ are constants. Equation (3-13) is employed by the analogy of the propagation law of creep cracks.

In the same derivation as that for the expression for the steady-state creep rate [45], $d\varepsilon/dn$ is given by

$$\frac{d\varepsilon}{dn} = w(S)A(S)n(S)b \ , \qquad (3\text{-}14)$$

according to the dislocation multiplication model proposed in the previous chapter. $w(S)$ denotes the dislocation multiplication frequency of a multiplication source per cycle, $n(S)$ the number of dislocation segments per unit volume which can be multiplied, $A(S)$ the area that multiplied dislocation sweeps out and $b$ the absolute value of Burgers vector. Taking into account the fact that the multiplied dislocations are piled up to cell-walls, $A(S)$ must be equal to $\{d(S)\}^2$. From probabilistic methods which are the same as those given in section 2-3, $w(S)$, $A(S)$, $n(S)$ and $d(S)$ are given by

$$n(S) = \frac{\kappa d_0}{l_0}\exp\left(-\frac{\delta\mu b}{l_0 S}\right) \ , \qquad (3\text{-}15)$$

$$d(S) = \beta\left(\frac{1}{n(S)}\right)^{\frac{1}{3}} \ , \qquad (3\text{-}16)$$

$$w(S) = \exp\left(-\frac{Q}{\sigma}\right) \ , \qquad (3\text{-}17)$$

and

$$A(S) = \{d(S)\}^2 = \beta^2\{n(S)\}^{-\frac{2}{3}} , \tag{3-18}$$

where

$$\sigma = \frac{c_0 S^2}{E} \cdot \frac{1}{n(S)} , \tag{3-19}$$

$c_0$ is a constant, $E$ elastic modulus, $\delta$ and $\beta$ are constants of order one, $\kappa$ denotes the probability that a dislocation-segment has an adequate configuration to glide under the external stress, $d_0$ the dislocation density in the early stage, $\mu$ rigidity modulus and $Q$ the activation energy. Substituting equations (3-15)-(3-19) into equation (3-14), we obtain

$$\frac{d\varepsilon}{dn} = \beta^2 b \left(\frac{\kappa d_0}{l_0}\right)^{\frac{1}{3}} \exp\left(-\frac{\delta\mu b}{3 l_0 S}\right) \exp\left(\frac{-Q}{\frac{c_0 S^2}{E} \cdot \frac{l_0}{\kappa d_0} \exp\left(\frac{\delta\mu b}{l_0 S}\right)}\right). \tag{3-20}$$

If $d\varepsilon/dn$ is plotted as a function of $S$ according to equation (3-20), it seems that equation (3-11) holds, and equations (3-12) and (3-20) yield

$$J^{(N)} = f(m)\beta^2 b\left(\frac{\kappa d_0}{l_0}\right)^{\frac{1}{3}} \exp\left(-\frac{\delta\mu b}{3l_0 S} - \frac{Q}{\frac{c_0 S^2}{E}\cdot\frac{l_0}{\kappa d_0}\exp\left(\frac{\delta\mu b}{l_0 S}\right)}\right) Sa. \qquad (3\text{-}21)$$

Furthermore, integrating equation (3-13), we can obtain the life of Stage $\mathrm{II_a}$, $N_{II_a}$, i.e. crack propagation life from grain-sized crack initiation to the occurence of crack with a length of $a_c$.

$$N_{II_a} = \int_{\gamma d_g}^{a_c} \frac{da}{a}\left(c_4 S\frac{d\varepsilon}{dn}\right)^{-1} = \left(c_4 S\frac{d\varepsilon}{dn}\right)^{-1} \ln\left(\frac{a_c}{\gamma d_g}\right), \qquad (3\text{-}22)$$

where $\gamma d_g$ is the depth of the crack whose surface length is equal to $d_g$, and $\gamma$ a constant.

## 3-3 Comparison with Experimental Data

In chapter 2, we obtain the life up to crack initiation with the use of a dislocation multiplication model. Following the procedure given there, the values of the parameters are determined and the theoretical results, using these values, are compared with those from experiments using 0.04% carbon steel with the grain size $d_g$=25μm[47]. The chemical composition and mechanical properties of the material are given in Table 3-1 and 3-2, respectively. The fatigue tests are performed for smooth specimens by a plane bending testing machine with stress ratio $R=-1$. In this experiment, the propagation rate of small fatigue cracks cannot be regarded as a function of stress intensity factor range. Carrying out the same procedure as in section 2-4 with the use of the data on the static tension test, we obtain $U/Q=0.275$, i.e. the ratio of the activation energy $Q$ to the threshold value for accumulated damage, $U$. Moreover, we put $\beta=1$, $\gamma=1$, $\kappa d_0/\delta=10^5\mathrm{mm}^{-2}$, $l_0/\delta=0.106\mu\mathrm{m}$ and $(Q/c_0)\cdot(\kappa d_0/l_0)=3.56\mathrm{MPa}$. Consequently, from equations (2-20) and (2-9), we obtain the average life up to grain-sized crack initiation $\overline{N}_{gi}$ and average life up to cell-sized crack initiation $\overline{N}_{ci}$. $\overline{N}_{gi}$ and $\overline{N}_{ci}$ are plotted in Fig. 3-1. Our theory well reproduces the experimental results for S-$\overline{N}_{gi}$ curves, and cell-sized crack

Table 3-1 Chemical composition of material used in reference [47]. (%)

| C | Si | Mn | P | S |
|---|---|---|---|---|
| 0.042 | 0.04 | 0.34 | 0.09 | 0.015 |

Table 3-2 Mechanical properties of material used in reference[47].

| Yield strength $S_Y$(MPa) | Tensile strength $S_B$(MPa) | Fatigue limit $S_w$(MPa) | Elongation (%) | Reduction of area (%) |
|---|---|---|---|---|
| 173 | 294 | 162 | 66.0 | 81.7 |

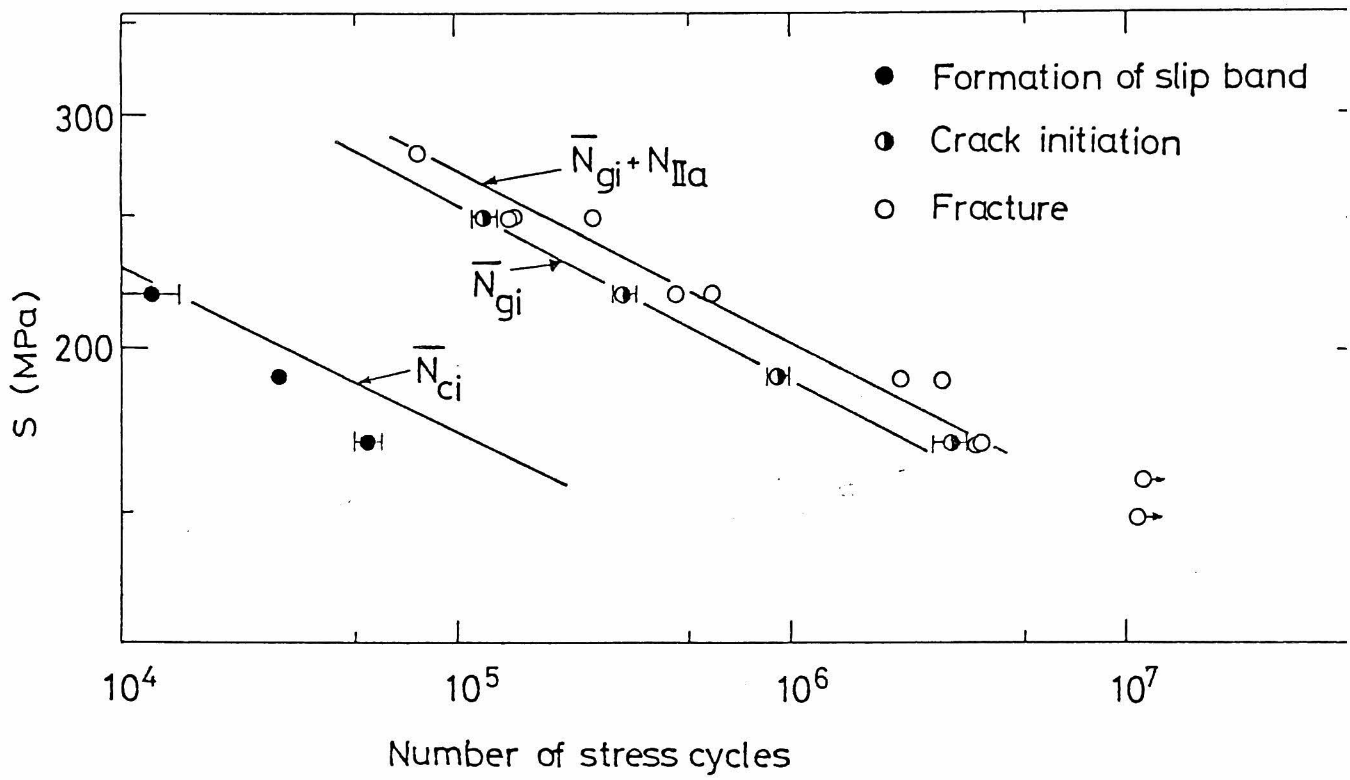


Fig. 3-1 Fatigue lives up to cell-sized crack initiation, grain-sized crack initiation and the end of Stage $II_a$ according to the present theory compared with experimental data[47]. (○, ◑, ●)

initiation occurs on the average after the formation of slip band, so our theory does not conflict with experimental results. If we choose $\delta=1$ and $\kappa=1$, we have $l_0=0.106\mu$m and $d_0=10^5$mm$^{-2}$. These values are adequate.

Now, substituting the values of parameters determined above into equation (3-14), we can calculate $d\varepsilon/dn$. In Fig. 3-2, we plot the experimental results $da/dn$ for small fatigue cracks as a function of $\frac{d\varepsilon}{dn}Sa$ which is in proportion to $J^{(N)}$. From Fig.3-2, we can conclude that the propagation rate of small fatigue cracks, $da/dn$, has the following relation,

$$\frac{da}{dn} \propto \frac{d\varepsilon}{dn}Sa, \tag{3-23}$$

i.e. it shows that equation (3-13) gives an adequate propagation law of small fatigue cracks. Furthermore, if we put

$$\frac{1}{c_4}\ln(a_c/\gamma d_g) = 14.76MPa,$$

we can calculate $N_{II_a}$, i.e. the crack propagation life of Stage II$_a$, from equation (3-22). The result $\overline{N}_{gi}+N_{II_a}$ is also plotted in Fig.3-1. $\overline{N}_{gi}+N_{II_a}$ is approximately equal to the fatigue life up to fracture.

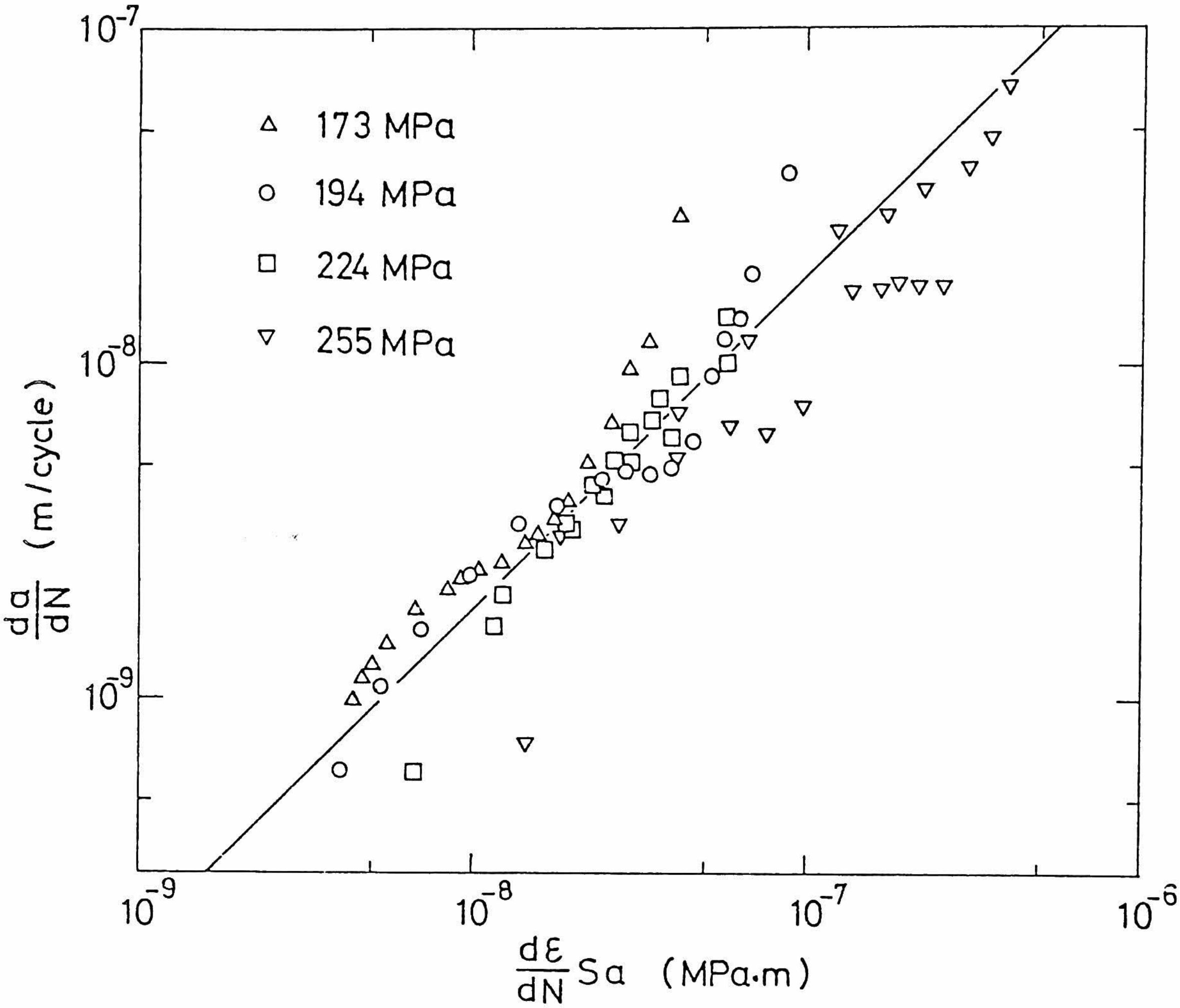


Fig. 3-2 Experimental results[47] for $da/dn$ plotted as a function of $\frac{d\varepsilon}{dn}Sa$. Full curve represents equation (3-13).

## 3-4 Some Remarks

(1) In this chapter we propose a modified J-integral (fatigue J-integral) $J^{(N)}$, and showed that the propagation rate of small fatigue cracks is proportional to $J^{(N)}$ by a comparison with the experimental results.

(2) From the foregoing propagation law, we calculated the life of Stage II$_a$, $N_{II_a}$. Adding it to the average life up to grain-sized crack initiation $\overline{N}_{gi}$ which is obtained with the use of the model proposed in the previous chapter, we showed that the estimation of total fatigue life is possible.

(3) From the remarks in the previous chapter and in this chapter, we can conclude that fatigue crack initiation and propagation of small cracks can be treated on the basis of the same dislocation multiplication model.

## Appendix 3-1 The Definition and Physical Meaning of the J-Integral

If the plastic region near a crack tip is so small that linear elastic fracture mechanics can be applied, the stress field near the crack tip is not affected by the plastic deformation at the crack tip and characterized by stress intensity factor $K$[15]. For large scale yielding crack problems, however, $K$ is not so effective[15]. So another parameter which better describes such cases needs to be introduced.

The path independent J-integral derived independently by Eshelby[48] and by Rice[49] is generally recognized to be a useful parameter that characterizes the stress field near the crack tip due to stationary cracks in elastic media. More recent studies have demonstrated that this integral parameter provides not only an accurate characterization of the crack tip elastic-plastic field but also a good elastic-plastic fracture criterion.

Here, we describe the definition of the J-integral. Consider a homogeneous body subjected to a two-dimensional deformation field so that all stresses $\sigma_{ij}$ depend only on two Cartesian coordinates $x$ and $y$. Suppose the body contains a crack having flat surfaces parallel to the x-axis. Define the J-integral $J$ by

$$J = \int_{\Gamma} (W dy - T_i \frac{\partial u_i}{\partial x} ds). \tag{3-24}$$

Here, $\Gamma$ is a curve surrounding the crack tip, and the integral is evaluated in a counterclockwise sense starting from the lower flat crack surface and continuing along the path $\Gamma$ to the upper flat surface. $T_i$ is the tension vector perpendicular to $\Gamma$ in an outside direction,

$$T_i = \sigma_{ij} n_j ,$$

where $n_j$ is the unit outward normal vector. $u_i$ is the displacement vector and $ds$ is an element of arc length along $\Gamma$. Furthermore,

$$W = W(x,y) = W(\varepsilon) = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij} ,$$

where $\varepsilon=[\varepsilon_{ij}]$ is the infinitesimal strain tensor, and $W(\varepsilon)$ the strain energy per unit volume. To prove path independency, consider any closed curve $\Gamma^*$ in a two-dimensional deformation field free of body forces. An application of Green's theorem yields

$$\int_{\Gamma^*} (W dy - T_i \frac{\partial u_i}{\partial x} ds) = 0, \tag{3-25}$$

for any closed curve $\Gamma^*$. Consider any paths $\Gamma_1$ and $\Gamma_2$ surrounding the notch tip. Traverse $\Gamma_1$ in the counter-clockwise direction, continue along the upper flat crack surface to the point where $\Gamma_2$ intersects the crack, traverse $\Gamma_2$ in the clockwise direction, and then continue along the lower flat crack surface to the starting point where $\Gamma_1$ intersects the crack. This describes a closed contour so that the integral $W dy - T_i (\frac{\partial u_i}{\partial x}) ds$ vanishes. But, $T=0$ and $dy=0$ on the portions of path along the flat crack surface. Thus the integral along $\Gamma_1$ counterclockwise and the integral along $\Gamma_2$ clockwise sum to zero. $J$ has the same value when computed by integrating along either $\Gamma_1$ or $\Gamma_2$, and path independency is proven. We assume, of course, that the area between curves $\Gamma_1$ and $\Gamma_2$ is free of singularities. Since $J$ is path independent, it can be determined in the easiest way by selecting a path along which the integration can be carried out conveniently.

Rice[49] pointed out that in general

$$J = -\frac{\partial V}{\partial a}, \tag{3-26}$$

where $V$ is the potential energy and $a$ a crack length, i.e. that $J$ is the energy release rate.

Amazigo[50] obtained the analytical expression of J-integral for the semi-infinite body with an edge crack of depth $a$, subjected to remotely applied shearing stress $\tau$. He considered a pure hardening stress-strain relation given by

$$\frac{\gamma}{\gamma_0} = \alpha\left(\frac{\tau}{\tau_0}\right)^m ,$$

where $\alpha$ is a nondimensional constant, $\gamma$ and $\tau$ are shear strain and stress, $\gamma_0$ and $\tau_0$ the reference values, and $m$ is the power hardening parameter. The obtained expression for $J$ is

$$J = a\tau\gamma f_1(m) , \tag{3-27}$$

where $f_1(m)$ is a definite function of $m$.

The J-integral under plane strain is calculated numerically and analogous results are obtained[43],

$$J = f(m)Sa\varepsilon , \tag{3-28}$$

where $S$ and $\varepsilon$ are stress and strain, and $f(m)$ is a function of $m$ which is obtained numerically. $f(m)$ is approximately given[43] by

$$f(m) = 3.85\sqrt{m}(1 - \frac{1}{m}) + \frac{\pi}{m} . \tag{3-29}$$

# CHAPTER 4

# A DAMAGE ACCUMULATION MODEL FOR ELASTO-PLASTIC FATIGUE

## 4-1 Introduction

In the previous chapters, we have investigated fatigue phenomena with the aid of a damage accumulation model. Hitherto, we have been concerned with high-cycle fatigue phenomena in which plastic deformation can be neglected, obtained life up to crack initiation for smooth specimens, and moreover, a propagation law of small fatigue crack. These results were in general agreement with the experiments.

We direct our attention to the elasto-plastic fatigue for smooth specimens. In other words, we investigate the fatigue phenomena in the case where plastic strain is the major part of total strain. In high-cycle fatigue in which plastic deformation can be neglected, we need not consider plastic effects, and assume that the damage accumulation process is generated by a Poisson process because of infrequent occurrence of damage accumulation. Furthermore, in this case, the number of places where damage accumulation occurs is small. In elasto-plastic fatigue these respects must be examined. We improve the damage accumulation model proposed in previous chapters in order that elasto-plastic fatigue phenomena can be treated, and calculate life up to crack initiation. The results are compared with experimental data[51],[18].

We see that our theory gives results that agree with those of the experiments. Moreover, with the use of the propagation law of small fatigue cracks proposed in chapter 3, the fatigue crack propagation life of Stage $II_a$ is calculated. Adding the result to life up to grain-sized crack initiation, we estimate fatigue life up to fracture. We then examine whether or not the damage accumulation model is applicable to elasto-plastic fatigue.

In section 4-2, the crack initiation model proposed in chapter 2 is examined. Section 4-3 comprises the investigation of a propagation law of small cracks. We make further remarks in section 4-4.

## 4-2 A Crack Initiation Model

In this section, we explain the outline of our model proposed in order to explain the crack initiation in high-cycle fatigue in which plastic effects can be neglected, and investigate improvements in treating elasto-plastic fatigue phenomena.

When an external repeated stress (stress amplitude $S$ and angular frequency $\omega$) is applied to a material, the energy $Y$ which is transiently stored in the material per active dislocation source per unit cycle is assumed to be a random variable with exponential distribution whose mean is $\sigma$, i.e.

$$Pr.\{Y \leqq y\} = H(y) = \begin{cases} 1 - \exp(-\frac{y}{\sigma}) & y \geqq 0 \\ 0 & y \leqq 0 \end{cases} \tag{4-1}$$

Only a part of the transiently stored energy beyond a certain amount $Q$ is accumulated as the deformation energy due to dislocation multiplication. This energy contributes to fatigue damage. This accumulated damage can be considered to cause crack formation. In other words, the strain energy $Z=\max(Y-Q,0)$ is accumulated inside the material in every cycle. It is assumed that the cell-sized crack nucleates when the accumulated damage exceeds a specified value $U$ for the first time. In high-cycle fatigue in which plastic deformation can be neglected, the event $Y>Q$ takes place infrequently, so that we consider that the event occurs according to a Poisson process. If so, the average fatigue life in the case that damage can decrease due to the mutual annihilation of dislocations can also be obtained for elasto-plastic fatigue, the assumption that the event $Y>Q$ occurs according to a Poisson process is not appropriate,because it seems that damage accumulation takes place frequently. But, if decay of fatigue damage is negligible, $N_{ci}$, the life up to cell-sized crack initiation, can be calculated without the assumption that the damage accumulation process is generated by a Poisson process. So, we can obtain the expression of $\overline{N}_{ci}$, the average of $N_{ci}$,

$$\overline{N}_{ci} = (1 + \frac{U}{\sigma})\exp(\frac{Q}{\sigma}) \quad , \tag{4-2}$$

which is the same form as before. A detailed derivation of equation (4-2) is given in Appendix 4-1. In elasto-plastic fatigue, the number of places where damage accumulation occurs is more than that in the high-cycle case. We say more about this later, where the number of active dislocation sources is discussed.

The cell-sized crack initiated in a material propagates until the surface length of the crack increases to $\xi$. $\xi$ is taken to be the initiated crack length measured in experiments. This propagation process is assumed as follows. When a cell-sized crack has initiated in a cell, the damage begins to accumulate in the adjacent cell and some further cycles are spent before the accumulated damage exceeds $U$ and the cell fails. Then the crack propagates across this cell. According to this manner , one cell after another fails until the crack length equals $\xi$. If we assume that the life from the beginning of damage accumulation in each cell up to failure of the cell is a mutually independent random variable with a common distribution, the life up to initiation of the crack with a length of $\xi$, $N_\xi$, is the summation of each cell's life from the beginning of damage accumulation up to failure. Consequently, the average of $N_\xi$, denoted by $\overline{N}_\xi$, is equal to $\overline{N}_{ci}$ multiplied by

$$\frac{a}{d(S)} = \frac{\gamma\xi}{d(S)} .$$

Here, we adopt the relation between crack depth $a$ and $\xi$,

$$a = \gamma\xi,$$

where $\gamma$ is a constant. That is to say,

$$\overline{N}_{\xi} = \overline{N}_{ci} \times \frac{\gamma\xi}{d(S)} ,$$

where $d(S)$ denotes the cell size. If $d(S)$ is proportional to the average mutual distance from one active dislocation source to the adjacent one, $d(S)$ is given by

$$d(S) = \beta\left(\frac{1}{n(S)}\right)^{\frac{1}{3}} , \tag{4-3}$$

where $\beta$ is a nondimensional constant, and $n(S)$ the number of active dislocation multiplication sources per unit volume. As stated above, in elasto-plastic fatigue, $n(S)$ may be much larger than that in high-cycle fatigue in which plastic deformation can be neglected. It can be argued, however, that the argument given in section 2-3 is also applicable to the elasto-plastic fatigue phenomena. Accordingly, if we examine the equilibrium condition between line tension of a dislocation and applied stress and assume that the length of a dislocation-segment $L$ is a random variable with an exponential distribution whose mean is $l_0$, $n(S)$ is given by

$$n(S) = \frac{\kappa d_0}{l_0}\exp\left(-\frac{\delta \mu b}{l_0 S}\right) , \qquad (4\text{-}4)$$

where $\kappa$ denotes the probability that a dislocation segment has an adequate configulation to glide under the external stress, $\delta$ a constant of order one, $\mu$ rigidity modulus, and $d_0$ the dislocation density in the early-stage. From the foregoing argument, $\overline{N}_\xi$ is obtainable if $\sigma$ is determined.

Now, we obtain the expression for $\sigma$. In high-cycle fatigue in which plastic deformation can be neglected, $\sigma$ is regarded as the value per one active dislocation multiplication source, and we put $\sigma=c_0S^2/E/n(S)$ where $E$ is an elastic modulus and $c_0$ a constant. $c_0S^2/E$ is the energy which stays inside the materials in one cycle per unit volume. Since, in high-cycle fatigue where plastic deformation can be neglected, the transiently stored energy can be regarded as the result of anelastic effects, $c_0S^2/E$ which is proportional to the area of hysteresis loop for the case of anelasticity is used there. In elasto-plastic fatigue, the effect of plastic deformation appears in the hysteresis loop, and its area is not proportional to $c_0 S^2/E$. In elasto-plastic fatigue, however, the stored energy per unit volume can also be considered to be proportional to the area of hysteresis loop. Consequently, it seems adequate to put $\sigma=c_0 B(S)/n(S)$ where $c_0$ denotes a constant and $B(S)$ the area of hysteresis loop.

From the above argument, we can discuss the S-$\overline{N}_{gi}$ curve, so we will compare our model with experimental data for S35C steel[51]. In this experiment, the fatigue tests for smooth specimens are perfomed under the stress amplitude which causes large plastic strain, and life up to initiation of the crack with a surface length of $\xi=50\mu$m is also measured. Then, we will decide the values of the parameters. First, we choose $U/Q=3.0$, since the values of $U/Q$ are about $3.0$ for all materials which we investigated in the previous chapters, and moreover, $U/Q$ has little effect on the final results. With the use of the experimental relation between strain range $\Delta\varepsilon$ and stress range $\Delta S$, we can obtain $B(S)$, i.e.

$$B(S) = (\frac{\Delta S}{k})^{4.27} \Delta S,$$

where $k=2.79$GPa. Furthermore, $\gamma=1/2$ is used. In the same way as we do for the high-cycle fatigue in which plastic deformation can be neglected, we choose $\beta=1$ and $\kappa d_0/\delta=1\times10^{11}\mathrm{m}^{-2}$. If we put $b\delta\mu/l_0=1200$MPa and $(\kappa d_0/l_0)\cdot(Q/c_0)=213.4$MPa, we can calculate $\overline{N}_{ci}$ and $\overline{N}_\xi$ from equations (4-2), (4-3) and (4-4). The results are shown in Fig.4-1. From Fig.4-1, we see that our model well reproduces the life up to initiation of the crack with a length of $\xi$. Moreover, if we put $\delta=1$ and $\kappa=1$, we obtain $d_0=1\times10^{11}\mathrm{m}^{-2}$ and $l_0=0.0173\mu$m. These values for the parameters seem to be proper. Next, we examine the fatigue experiment for S15C steel[18] which is conducted in a rotating bending fatigue testing machine. This experiment can be called a high-cycle fatigue test on account of the value of fatigue life, but plastic strain is very large in this fatigue test. The relation between $\Delta\varepsilon$ and $\Delta S$ is also measured in this experiment. The data yields

$$B(S) = \left(\frac{\Delta S}{k}\right)^{2.76} \Delta S ,$$

where $k=4.65$GPa. We put $\gamma=1/2$, $U/Q=3.0$, $\beta=1$ and $\kappa d_0/\delta=1\times10^{11}\mathrm{m}^{-2}$ which are the same values as for S35C, and we choose $\delta\mu b/l_0=550$MPa and $(\kappa d_0/l_0)\cdot(Q/c_0)=57.46$MPa. Using these values for the parameters, $\overline{N}_{ci}$ and $\overline{N}_\xi$ for $\xi=0.02$mm are obtained and the results are plotted in Fig. 4-2. Figure 4-2 shows that $\overline{N}_\xi$ is in agreement with the experimental data. Moreover, if we put $\kappa=1$ and $\delta=1$, we obtain $d_0=1\times10^{11}\mathrm{m}^{-2}$ and $l_0=0.0377\mu\mathrm{m}$. These values of $d_0$ and $l_0$ seem to be reasonable. From the argument stated above, it may be considered that our damage accumulation model is well applicable to elasto-plastic fatigue phenomena. In the next section, in order to reexamine this point, we will calculate the fatigue J-integral using the same values for the parameters as previously determined in this section. We will then derive an expression for the propagation rate of small fatigue cracks. Furthermore, calculating the fatigue life of Stage $\mathrm{II}_a$ by the use of this expression, we will estimate the total fatigue life.

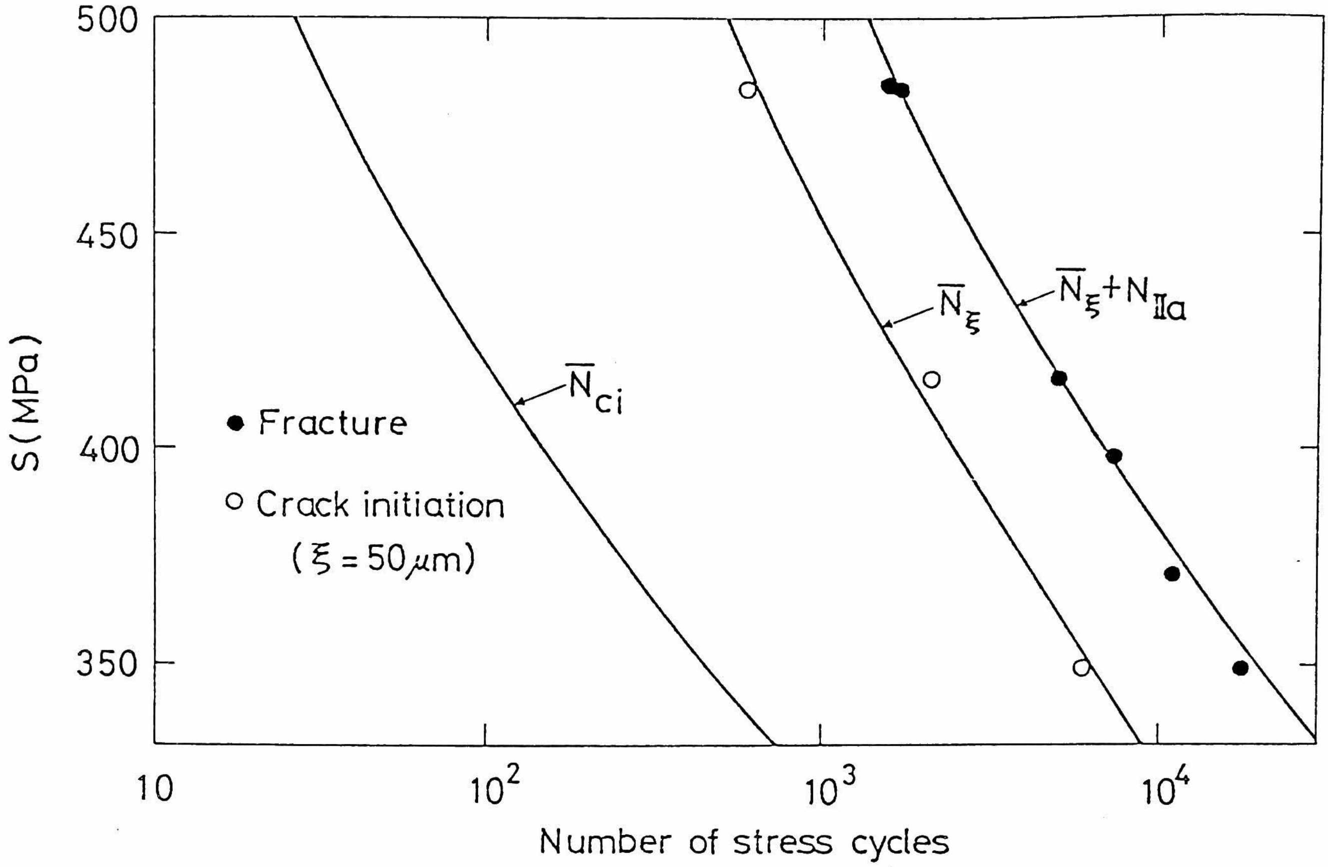


Fig. 4-1 Fatigue life up to initiation of a cell-sized crack, initiation of a crack with a length of $50\mu m$ and the end of Stage $II_a$ according to the present theory compared with experimental data for S35C (○,●). [51]

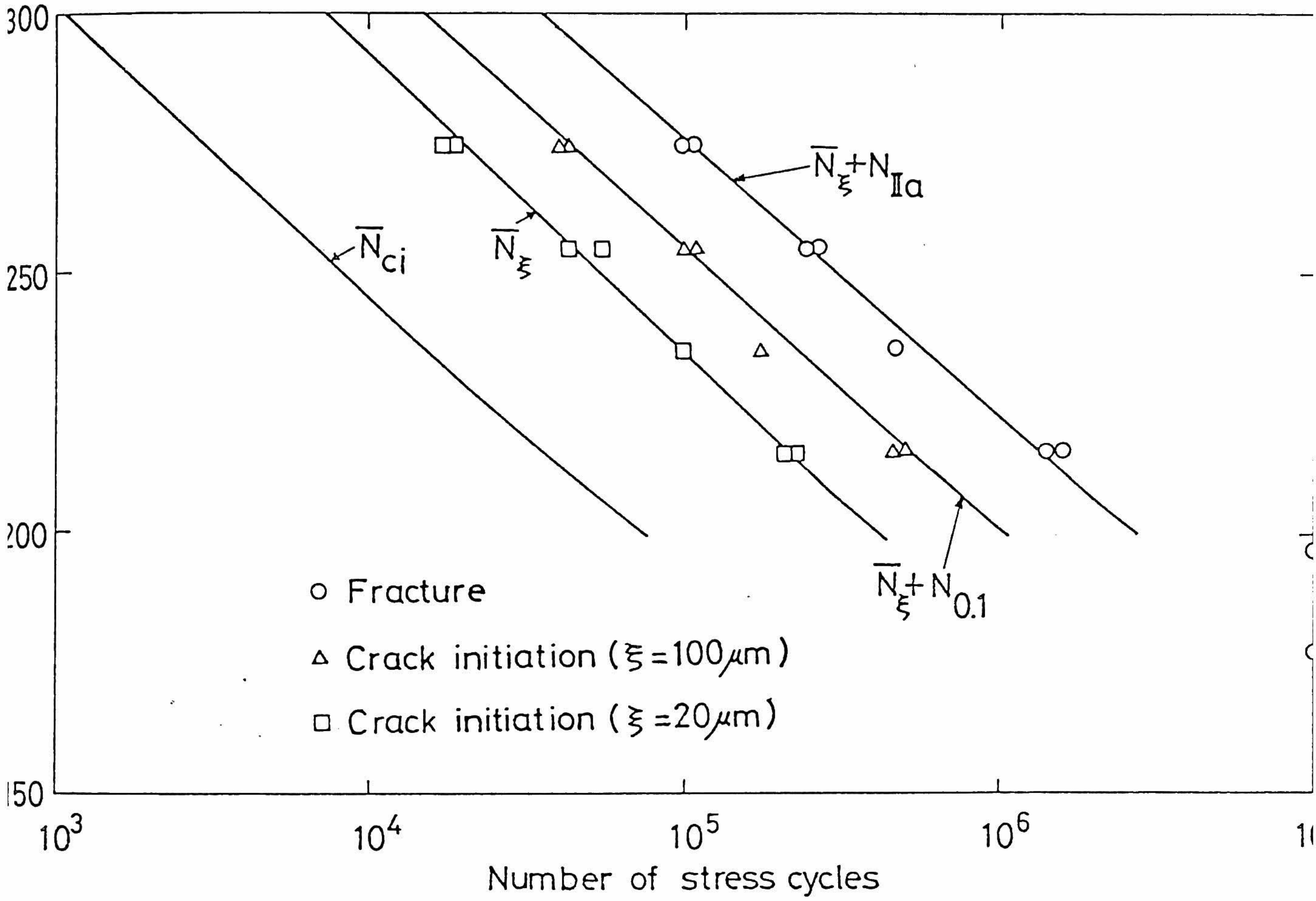


Fig. 4-2 Fatigue life up to initiation of a cell-sized crack, initiation of a crack with a length of *20*μm, initiation of a crack with a length of *100*μm and the end of Stage $II_a$ according to the present theory compared with experimental data for S15C (O,Δ,□). [18]

## 4-3 Derivation of a Propagation Law of Small Cracks

In the experiments for S15C[18], the propagation rate of small fatigue cracks after the initiation of crack with a length of $\xi=0.02$mm is measured, So, we investigate this crack growth by the use of the fatigue J-integral proposed in chapter 3. The calculation of the fatigue J-integral is performed according to the forementioned improved model for damage accumulation. As stated in chapter 3, the fatigue J-integral $J^{(N)}$ is given by

$$J^{(N)} = c_3 \frac{d\varepsilon}{dn} S a \ , \qquad (4\text{-}5)$$

where $a$ denotes the depth of a crack, $S$ the stress amplitude, $d\varepsilon/dn$ the increment of maximum strain per one cycle and $c_3$ a constant. In chapter 2, we showed that the propagation rate of small fatigue cracks is proportional to $J^{(N)}$ for 0.04% steel. As in chapter 2, $d\varepsilon/dn$ is given by

$$\frac{d\varepsilon}{dn} = w(S)A(S)n(S)b \ , \qquad (4\text{-}6)$$

which is derived on the basis of the dislocation multiplication model. Here, $w(S)$ denotes the dislocation multiplication frequency of a multiplication source per load cycle, and is given by

$$\exp(-\frac{Q}{\sigma}) \ ,$$

where we put $\sigma = c_0 B(S)/n(S)$ as in the last section, $A(S)$ is the area that the multiplied dislocation sweeps out and $b$ the absolute value of Burgers vector. $A(S)$ can be considered to be equal to the square of cell size $d(S)$, i.e.

$$A(S) = \{d(S)\}^2 \ . \qquad (4\text{-}7)$$

$d(S)$ and $n(S)$ are calculated from equations (4-3) and (4-4), respectively. From equations (4-5), (4-6) and (4-7), we obtain $\frac{d\varepsilon}{dn}Sa$ which is proportional to $J^{(N)}$. In Fig. 4-3, experimental data $da/dn$ for a small fatigue crack is plotted as a function of $\frac{d\varepsilon}{dn}Sa$, where $B(S)$ in the expression of $\sigma$ is calculated from the expression shown in the last section, and the values of all parameters which appear in equations (4-6) and (4-7) are taken to be equal to the values obtained in section 4-2. Figure 4-3 shows that $da/dn$ the crack propagation rate can be reasonably well

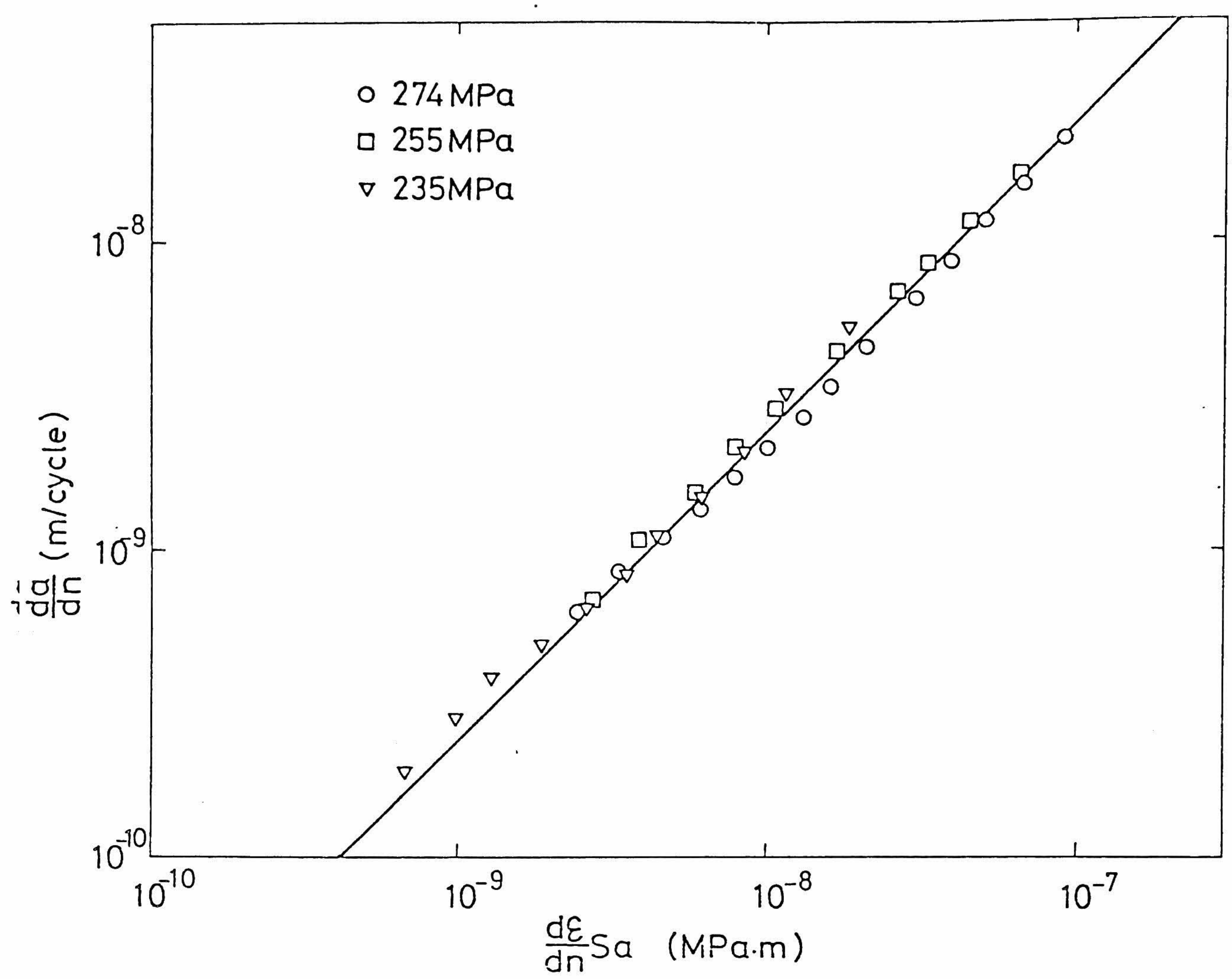


Fig. 4-3 Experimental results $da/dn$ for S15C[18] plotted as a function of $\frac{d\varepsilon}{dn}Sa$. Full curve represents equation (4-8).

approximated in this case by

$$\frac{da}{dn} = c_4 \frac{d\varepsilon}{dn} Sa, \tag{4-8}$$

where $c_4 = 0.24\mathrm{MPa}^{-1}$. That is to say, it is confirmed that the propagation rate of small fatigue crack for this experiment is also proportional to $J^{(N)}$ as in Chapter 3.

Suppose that, after the crack length attains $\xi$, the crack propagates according to equation (4-8), we obtain the life of Stage II$_a$ $N_{II_a}$, and the fatigue life for S15C from the initiation of the crack with a depth of $\gamma\xi$ to the stage where crack length attains to $a_c$ by integrating equation (4-8). $\overline{N}_\xi + N_{II_a}$ is shown in Fig.4-2 where we put $\frac{1}{c_4}\ln(a_c/\gamma\xi) = 20.03\mathrm{MPa}$. If we substitute $c_4 = 0.24\mathrm{MPa}^{-1}$, $a_c = 2.45\mathrm{mm}$ is obtained. $\overline{N}_\xi + N_{II_a}$ is quite near to the total fatigue life in Fig.4-2. If we assume that equation (4-8) holds for S35C (the value of $c_4$ may not be equal to $0.24\mathrm{MPa}^{-1}$), we can calculate $N_{II_a}$ for S35C. The results are shown in Fig.4-1 where we put $\frac{1}{c_4}\ln(a_c/\gamma\xi) = 18.42\mathrm{MPa}$. The values of the other parameters are taken to be equal to the values used in the last section.
In S15C steel, $\overline{N}_\xi + N_{II_a}$ is also approximately equal to the

life up to fracture. For S15C, the life up to initiation of the crack with a length of $0.1$mm is measured. Integrating equation (4-8), we calculate $N_{0.1}$, the life from initiation of the crack with a length of $\xi=0.02$mm to initiation of the crack with a length of $0.1$mm. $\overline{N}_{\xi}+N_{0.1}$ is shown in Fig.4-2, and is in general agreement with the experimental data. According to the above argument, we may conclude that, in elasto-plastic fatigue, crack initiation and the propagation law of small cracks can also be explained by using the damage accumulation model.

## 4-4 Some Remarks

In the previous chapters, we proposed a damage accumulation model for high-cycle fatigue in which plastic deformation can be neglected. In this chapter, the model has been improved in some respects in order that elasto-plastic fatigue phenomena can be treated. With the use of this improved model, life up to crack initiation and the propagation rate of small fatigue cracks in elasto-plastic fatigue are obtained, and compared with the experimental data. They tend to coinside. Consequently, the appropriateness of the model for elasto-plastic fatigue can be shown. Moreover, with the crack initiation model and with the propagation law of small cracks, the total fatigue life is calculated.

## Appendix 4-1 The Derivation of Equation (4-2)

The probability that $N_{ci}$, life up to cell-sized crack initiation is equal to $n$ is denoted by $f(n)$. $f(n)$ is given by

$$f(n)=Pr.\{N_{ci}=n\}=\int_0^U dxg(x,n-1)w(S)\exp(-\frac{U-x}{\sigma}) \quad , \tag{4-9}$$

where $g(x,n-1)dx$ is the probability that the damage accumulated during $n-1$ cycles is between $x$ and $x+dx$, and $w(S)$ the probability that the damage accumulation occurs in a load cycle. We put

$$w(S) = \exp(-\frac{Q}{\sigma}).$$

The left-hand side of equation (4-9) expresses the probability that $N_{ci}$ is $n$ load cycles, so that $0<x<U$. Moreover, $w(S)\exp\{-(U-x)/\sigma\}$ is the probability that the damage larger than $U-x$ is accumulated in the $n$'th cycle Consequently, the integral of the product of $g(x,n-1)$ and $w(S)\exp\{-(U-x)/\sigma\}$ from $0$ to $U$ is equal to $f(n)$.

Now, $g(x,n)$ is expressed as follows.

$$g(x,n) = \delta(x)P_n(0) + \sum_{l=1}^{n} P_n(l)g_l(x) \ , \tag{4-10}$$

where $P_n(l)$ is the probability that damage accumulation occurs $l$ times during $n$ load cycles, $g_l(x)dx$ the probability that the accumulated damage is between $x$ and $x+dx$ if the damage accumulation occurs $l$ times and $\delta(x)$ is Dirac's delta function. From the definition, the events, damage accumulation occurs in one cycle, are mutually independent. Since the probability that damage accumulation occurs in a load cycle is $w(S)$, $P_n(l)$ is the binomial distribution

$$P_n(l) = \binom{n}{l}\{w(s)\}^l\{1-w(S)\}^{n-l} \ . \tag{4-11}$$

$g_l(x)$ is the probability density function of the sum of $l$ mutually independent random variables with an exponential distribution whose mean is $\sigma$, so we have

$$g_l(x) = \frac{x^{l-1}}{\sigma^l(l-1)!}e^{-\frac{x}{\sigma}} \ . \tag{4-12}$$

From equations (4-9)-(4-12), we get

$$f(n) = w(S)e^{-\frac{U}{\sigma}} \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-l-1)!l!} \{w(S)\}^{l}$$

$$\times \{1 - w(S)\}^{n-l-1} (\frac{U}{\sigma})^{l} .$$

(4-13)

From equation (4-13), $\overline{N}_{ci}$ is given by

$$\overline{N}_{ci} = \sum_{n=1}^{\infty} nf(n) = w(S)e^{-\frac{U}{\sigma}} \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \frac{n!}{(l!)^2(n-l-1)!} \{\frac{U}{\sigma} w(S)\}^{l} \{1 - w(S)\}^{n-l-1}$$

$$= w(S)e^{-\frac{U}{\sigma}} \sum_{l=0}^{\infty} \frac{1}{(l!)^2} (\frac{U}{\sigma} w(S))^{l} \sum_{n=l+1}^{\infty} \frac{n!}{(n-l-1)!} \{1 - w(S)\}^{n-l-1}$$

Using the relation

$$\frac{1}{(1-x)^m} = \sum_{j=0}^{\infty} \frac{(m+j-1)!}{j!(m-1)!} x^{j}$$

to the summation of $n$, we obtain

$$\overline{N}_{ci} = w(S)e^{-\frac{U}{\sigma}} \sum_{l=0}^{\infty} \frac{l+1}{l!} (\frac{U}{\sigma})^{l} \frac{1}{\{w(S)\}^2}$$

$$= \frac{1}{w(S)} e^{-\frac{U}{\sigma}} (\frac{U}{\sigma} e^{\frac{U}{\sigma}} + e^{\frac{U}{\sigma}})$$

$$= (1 + \frac{U}{\sigma}) e^{\frac{Q}{\sigma}} . \qquad (4\text{-}14)$$

# CHAPTER 5

# A STOCHASTIC MODEL FOR THROUGH CRACK GROWTH UNDER REPEATED STRESS

## Introduction

It is well established experimentally that under repeated stress the propagation rate of a through crack is proportional to some power of the stress intensity range $\Delta K$ in a suitable range of $\Delta K$[52].

$$\frac{da}{dn} = c(\Delta K)^{\nu}, \tag{5-1}$$

where $da/dn$ denotes crack growth rate, and $c$ and $\nu$ are constants. For most metals, the experimental values for the exponent $\nu$ are between $2$ and $4$. There are some theoretical approaches to derive this power law [53]-[56]. In these theories, the exponent has the same value for all materials ($2$ or $4$) although various values of the exponent are obtained experimentally.

Recently Oh[26] has proposed a weakest link model to explain this variation of the exponent for crack propagation rate. He assumed as follows: The region near the crack tip is composed of a chain of material elements, and under repeated stress fatigue damage accumulates inside each element. When this damage accumulation in a certain element exceeds the element's strength, the element fails and the crack propagates to the element. Oh further assumed that the strength of an element is a random variable, and that the asymptotic form of its distribution function is in proportion to some power of the strength. Based on these assumptions, he derived the power law for the crack propagation rate. However, Oh's model uses some debatable assumptions: the load cycle $N$ expended for one step of crack growth is treated as a continuous random variable and the distribution function of the strength of an element is assumed to be proportional to the element's length, which is allowed to approach zero. Consequently, Oh's model results in the divergence of the crack propagation rate under certain conditions.

In this chapter, we reexamine Oh's model closely, and derive an improved expression for the rate of subcritical growth of fatigue cracks. Furthermore, using the distribution of the life of material elements which are obtained on the basis of a dislocation multiplication model proposed in previous chapters, we calculate the propagation rate and compare it with experiments[31].

In Part A, Oh's model is improved in some respects and the power law of crack propagation which is free from the difficulty of the divergence mentioned above is derived. In Part B, the expression for the fatigue crack propagation rate is derived from a two-dimensional version of the stochastic model proposed in Part A, and the distribution function of the life of a material element in this expression is calculated with the aid of the dislocation multiplication model. Our results are compared with the experimental data.

## Part A

## A Probabilistic Approach to Fatigue Crack Growth Rate

### 5-A-1 Introduction

The aim of Part A is to improve Oh's model for crack propagation[26] in order to extend the applicability range of his model. First, we treat the load cycles $N$ required for growth as a discrete random variable so that the theory can be applied to the striation regions in which Oh's theory does not work. By so doing, we can avoid the divergence appearing in Oh's treatment. Secondly, we introduce a measure of relative ductility, namely the ratio of the ductility to the damage accumulation rate. Assuming that the nearer to the crack tip the element is, the smaller is the ductility of an element, we can take into account the damage accumulation during the previous growth. Thirdly we fix a scale of the material elements so that we can adopt a natural form of probability distribution for the relative ductility.

Under the modifications above, results analogous to Oh's are deduced except that the exponent has a more complicated relation to the parameter of the distribution function than Oh's. Further, it is shown that the power law holds approximately even for a range of small $\langle N \rangle$.

In section 5-A-2, we specify our probabilistic model. The model is analyzed in section 5-A-3. In section 5-A-4, we show our numerical results and make some further remarks.

## 5-A-2 A Probabilistic Model

Following Oh's idea, we investigate the damage accumulation process in the crack tip region. In order to formulate the problem, we first assume that the crack tip region is composed of a chain of small material elements, each of which has constant scale $d_0$. Counting from the crack tip, we give these elements number $1, 2, \cdots, l, \cdots, l_0$, and then consider the ratio $Y_l$ of the ductility to the damage accumulation rate in the $l$'th element, which we call the relative ductility. It can be assumed that the $Y_l$'s are random variables whose averages depend on $l$, for the ductility increases with $l$ due to the damage accumulation under the previous growth, while the damage accumulation rate decreases with $l$ due to the stress distribution in the tip region Suppose that the relative ductility is measured with a unit of load cycles, for example, an event $n-1<Y_l<n$ indicates that the $l$'th element has a relative ductility that can endure for $n-1$ load cycles but cannot for $n$ load cycles.

Now, we will introduce new random variables $N$ and $M_n$ as follows:

$$N=\{n \mid n-1<Y_l \text{ for all } l \text{ and } Y_l \leq n \text{ for at least one } l\}, \tag{5-2}$$

$$M_n=\{m \mid n-1<Y_l \text{ for } l=1,2,\cdots,m-1,\ n-1<Y_m\leq n \text{ and}$$
$$n<Y_l \text{ for } l=m+1,m+2,\cdots,l_0\}, \qquad (5\text{-}3)$$

i.e. $M_n$ denotes the maximum element number which fails between $n-1$ and $n$ load cycles. Then we will define the crack growth rate as a random variable $d_0M_n/N$ and identify $\langle d_0M_n/N\rangle$ with $da/dn$. We will introduce a characteristic region $\omega$ in the crack tip, outside of which the damage accumulation is little significance. If we get $\omega$-dependence of $da/dn$, we can know how the crack growth rate behaves with the stress intensity factor range $\Delta K$, since the relation

$$\omega = \kappa\left(\frac{\Delta K}{2S_y}\right)^2 \qquad (5\text{-}4)$$

holds, where $\kappa$ denotes an adequate numerical constant and $2S_y$ the dynamical yield stress.

### 5-A-3 Analysis

We investigate the $\omega$-dependence of the mean value of the random variable $d_0 M_n/N$. Putting

$$Pr.\{Y_l \leq y\} = F_l(y), \tag{5-5}$$

from the definition (5-3) we get

$$\begin{aligned} Pr\{M_n = m,\ N = n\} &\equiv P(m,n) \\ &= \prod_{l=1}^{m-1} \bar{F}_l(n-1)\{F_m(n) - F_m(n-1)\} \prod_{j=m+1}^{l_0} \bar{F}_j(n), \end{aligned} \tag{5-6}$$

where $\bar{F}_l(y) \equiv 1 - F_l(y)$. We will discuss only the asymptotic behavior of this probability as the size $l_0$ tends to infinity. To this end, we can treat the $F_l(y)$'s as small variables. So, equation (5-6) is written as

$$\begin{aligned} P(m,n) \simeq \{F_m(n) - F_m(n-1)\}\exp\{&-\sum_{l=1}^{m-1} F_l(n-1) \\ &- \sum_{l=m+1}^{\infty} F_l(n)\} . \end{aligned} \tag{5-7}$$

As we need the form of $F_l(y)$ only for small $F_l(y)$, we assume that for small $F_l(y)$

$$F_l(y) = \begin{cases} \lambda(l)y^{\beta} & (y \geq 0) \\ 0 & (y < 0) \end{cases}, \tag{5-8}$$

where $\beta$ is a positive parameter and $\lambda(l)$ a very small decreasing function of $l$. From equations (5-7) and (5-8) we have approximately

$$P(m,n) \simeq \lambda(m)\{n^{\beta}-(n-1)^{\beta}\}\exp\{-(n-1)^{\beta}\sum_{l=1}^{m-1}\lambda(l)-n^{\beta}\sum_{l=m+1}^{\infty}\lambda(l)\} \tag{5-9}$$

$$\simeq\lambda(m)\{n^{\beta}-(n-1)^{\beta}\}\exp\{-(n-1)^{\beta}\int_0^m\lambda(l)dl-n^{\beta}\int_m^{\infty}\lambda(l)dl\} .$$

We note that the distribution of $N$ is easily obtainable from equation (5-9) within this approximation and reduces to a discrete version of a Weibull distribution[57], namely for $n=1,2,3,\cdots$

$$P(n)=\{\exp(-\int_0^{\infty}\lambda(l)dl)\}^{(n-1)^{\beta}}-\{\exp(-\int_0^{\infty}\lambda(l)dl)\}^{n^{\beta}} , \tag{5-10}$$

with parameters $0<\beta$ and $0<\exp\{-\int^{\infty}\lambda(l)dl\}<1$, and that if $\beta\neq 1$ the random variables $N$ and $M_n$ are not mutually independent.

In order to clarify our discussion, we fix the form of $\lambda(l)$ as follows:

$$\lambda(l) = \alpha\exp(-\frac{d_0}{\omega}l) , \qquad (5\text{-}11)$$

where $\alpha$ is a very small dimensionless positive constant. This is the simplest expression that has little significance outside the characteristic region $\omega$. Then, from equation (5-9), we have

$$\begin{aligned} \frac{da}{dn} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{d_0 m}{n}P(m,n) \\ &\simeq \frac{d_0}{\alpha}q\sum_{n=1}^{\infty}\frac{x_n}{n}\exp\{-(n-1)^{\beta}q\}\int_0^{\infty}\xi\exp(-\xi-x_n e^{-\xi})d\xi \\ &= \frac{d_0}{\alpha}q\sum_{n=1}^{\infty}\frac{1}{n}\exp\{-(n-1)^{\beta}q\}(\gamma+\ln x_n-E_i(-x_n)\}, \end{aligned} \qquad (5\text{-}12)$$

where

$$q \equiv \alpha\frac{\omega}{d_0},$$

$$x_n \equiv \{n^{\beta} - (n-1)^{\beta}\}q,$$

$$E_i(-x) \equiv -\int_x^{\infty}\frac{e^{-t}}{t}dt,$$

and $\gamma$ is the Euler's constant. We may understand that $q$ is a dimensionless quantity related to the plastic zone size. Hence owing to equation (5-4) the relation

$$q = \left(\frac{\Delta K}{\Delta K_0}\right)^2 \tag{5-13}$$

holds, so with the aid of equation (5-12) we can study the $\Delta K$-dependence of the crack growth rate.

### 5-A-4 Results and Remarks

In Fig.5-1, we show the increase in dimensionless average crack growth rate $\frac{\alpha}{d_0} \cdot \frac{da}{dn}$ with $\Delta K/\Delta K_0$ for various values of $\beta$. From Fig.5-1, we see that the curves obtained behave as if the relation (5-1) held for the parameter range measured by the experiments and that if we assume the power law (shown in equation (5-1)), the exponent $\nu$ ranges from *2* to *4*, as has been found in many experiments. Strictly, however, $\Delta K$-dependence of $da/dn$ is very complex, for example, if $\beta=1$ we have the exact expression

$$\frac{\alpha}{d_0}\frac{da}{dn} = -qe^{q}\{\gamma+\ln q-E_i(-q)\}\ln(1-e^{-q}). \tag{5-14}$$

Our model shows that a crack grows by $d_0M_n$ at $N$ cycles. Therefore, it is of interest to see how $<N>$ depends on $\Delta K/\Delta K_0$. From equations (5-10) and (5-11), $<N>$ reduces to

$$<N> = \sum_{n=1}^{\infty} \exp\{-(n-1)^{\beta}q\} , \tag{5-15}$$

which is plotted versus $\Delta K/\Delta K_0$ in Fig.5-2. Figure 5-2 shows that the behavior of $\langle N \rangle$ depends greatly on $\beta$ and that if $\beta$ is large enough, $\langle N \rangle$ has little dependence on $\Delta K$. Nevertheless, $da/dn$ for each $\beta$ behaves in a similar fashion. This is a very interesting feature.

Some investigators do not accept the damage accumulation hypothesis for fatigue crack growth due to the fact that the crack growth rate increases transiently as a result of intermediate reheat treatment to remove fatigue damage[58] and it decreases as a result of pre-straining[59]. Since we adopt a kind of relative ductility, however, the above treatments act upon not only its ductility but also the damage accumulation rate at that time. The latter may depend on the instantaneous values such as crack opening displacement, dislocation structures in the tip and so on. So the experimental results above seem not to be inconsistent with our model.

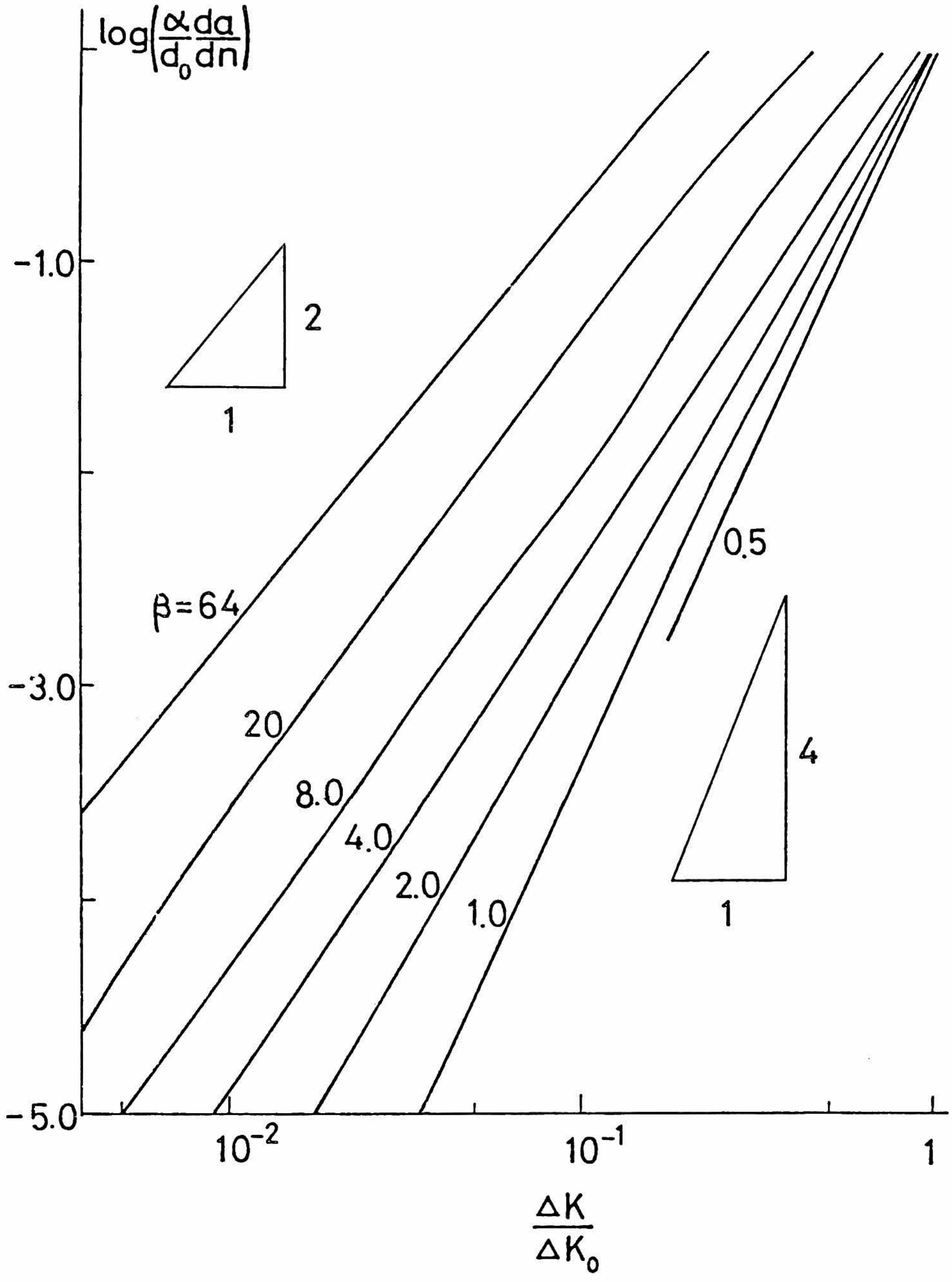


Fig. 5-1 Plot of $\log(\frac{\alpha}{d_0}\frac{da}{dn})$ versus $\log(\Delta K/\Delta K_0)$.

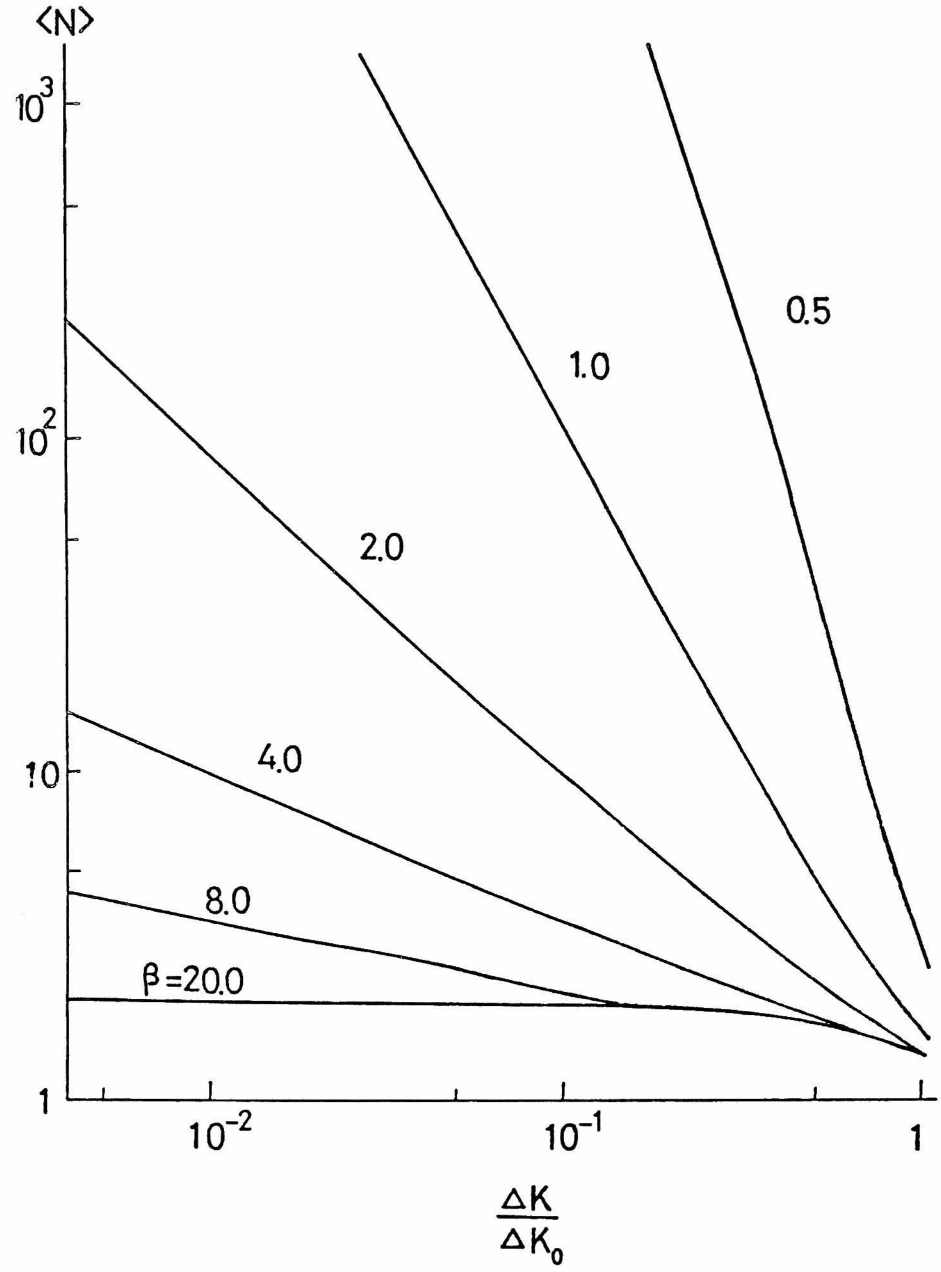


Fig. 5-2 Plot of log$\langle N \rangle$ versus log$(\Delta K/\Delta K_0)$.

## Part B

## A Damage Accumulation Model for Fatigue Crack Propagation

### 5-B-1 Introduction

In Part A in order to modify the faults of Oh's model, we proposed a stochastic model for crack propagation in which the load number is treated as a discrete random variable and the element's length has a fixed finite value. In the proposed model, however, the form of distribution function is not derived from any physical arguments, but rather assumed a priori as Oh does. The distribution function of the life of each material element is derived from a dislocation multiplication model which was applied to the problem of the fatigue life up to crack initiation treated in chapter 2. Here, we consider the life of material elements instead of their relative ductility employed in Part A. Moreover, we select a two-dimensional version of the stochastic model proposed in Part A because the crack tip region where the fatigue damage is concentrated can be treated as being two-dimensional. Finally, we suppose that this region is a small part of the plastic zone near the crack tip.

From the foregoing arguments, it is shown that the fatigue crack propagation rate is proportional to some power of the stress intensity range and that the exponent has various values larger than 2. Our results are compared with experimental data[21].

In section 5-B-2. our model for through crack growth is explained, and an expression for crack growth rate is obtained with the use of a distribution function of lives of material elements. The form of the distribution function is derived in section 5-B-3. We calculate the crack growth rate and compare the results with the experiments in section 5-B-4. We make additional remarks in section 5-B-5.

### 5-B-2 A Crack Propagation Model

Under repeated stress, many dislocations are multiplied inside the plastic zone near the crack tip and many slip bands are formed. Inside the crack tip region, the accumulation of fatigue damage is particularly remarkable due to the mutual crossing of slip bands. The damage accumulated outside this region, hereafter called "region D", is small. As shown in Fig.5-3, we can treat the plastic zone which includes the region D as a two-dimensional configuration of small material elements, each of which has scale $d_0$. In Fig.5-3. the origin is taken to be the crack tip and the positive direction of the x-axis to be coincident with the principal direction of crack propagation. The z-axis is taken to cross the x-axis at right angles. An element which is at a distance of $id_0$ in the direction of the x-axis and $jd_0$ in the direction of the z-axis is called an element $(i,j)$ (where $i=1,2,\cdots,i_0$, $j=\pm 1,\pm 2,\cdots,\pm j_0$). We consider the case where the life of each element is a random variable. The life of element $(i,j)$ is denoted by $H(i,j)$ and its distribution function by $H_{ij}(h)$. That is, $H_{ij}(h)$ is defined by

$$H_{ij}(h) \equiv Pr.\{H(i,j) \leq h\}, \tag{5-16}$$

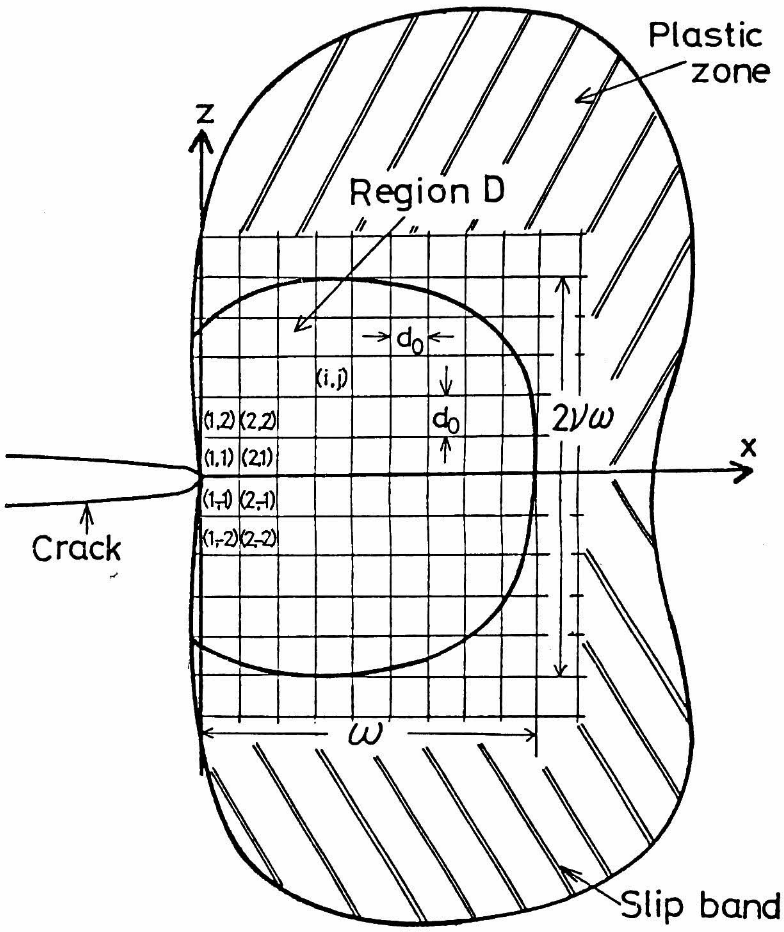


Fig. 5-3 Plastic zone and "region D" ahead of crack tip.

where Pr.$\{\mathcal{A}\}$ is the probability that an event $\mathcal{A}$ occurs. The life of an element is measured in terms of the load cycles which the element can endure. In section 5-B-3, we investigate $H_{ij}(h)$ in detail. Next, we refer to $2j_0$ elements $(i,\pm 1)$, $(i,\pm 2),\cdots,(i,\pm j_0)$ as the $i$'th column and give a number $i$ to the column. $Y(i)$, the life of the $i$'th column, is regarded as the life of the element that has the shortest life in the column. From equation (5-16), the distribution function of $Y(i)$, denoted by $F_i(y)$, can be expressed as

$$F_i(y) \equiv Pr.\{Y(i) \leq y\} = 1 - \prod_j (1 - H_{ij}(y)). \tag{5-17}$$

In order to investigate the crack propagation rate, we investigate two random variables $N$ and $M_n$, whose definitions are as follows:

$$N \equiv \{n \mid n-1 < Y(i) \text{ for all } i, \text{ and } Y(i) \leq n \text{ for at least one } i\}.$$

$$M_n \equiv \{m \mid n-1 < Y(i) \text{ for } i=1,2,\cdots,m-1,\ n-1 < Y(m) \leq n \text{ and } n < Y(i) \text{ for } i=m+1,\cdots,i_0\}, \tag{5-18}$$

where $N$ denotes a cycle number in which at least one column fails for the first time, and an event $n-1<Y(m)\leq n$

indicates that the $m$'th column can endure for $(n-1)$ load cycles, but not for $n$ load cycles. It is assumed that when some columns fail, the crack propagates to the farthest failing column from the crack tip in the direction of the x-axis and that when one column fails, the crack propagates instantaneously to the column. From this viewpoint, $N$ is the cycle number expended for one step of crack growth. $M_n$ represents the maximum column number that fails between $n-1$ and $n$ load cycles when $N=n$, so $d_0 M_n$ is the increment of crack length in a direction of the x-axis after $n$ cycles which are expended for one step of crack growth. Therefore the crack propagation rate can be defined as a random variable $d_0 M_n/N$ and its average $\langle d_0 M_n/N \rangle$ is identified with the crack propagation rate $da/dn$ observed in experiments

$$\frac{da}{dn} = \left\langle \frac{d_0 M_n}{N} \right\rangle , \tag{5-19}$$

where $a$ is a crack length. From equations (5-17) and (5-18), the joint probability of $M_n$ and $N$ is

$$P(m,n) \equiv Pr.\{M_n=m,\ N=n\} = \prod_{i=1}^{m-1} (1-F_i(n-1))\{F_m(n) - F_m(n-1)\} \prod_{i=m+1}^{i_0} (1-F_i(n)), \tag{5-20}$$

and the distribution of $N$ is

$$P(n) \equiv Pr.\{N=n\}=\prod_i (1-F_i(n-1))-\prod_i (1-F_i(n)). \qquad (5\text{-}21)$$

Therefore, from equation (5-20), we have

$$\frac{da}{dn}=\sum_{n=1}^{\infty}\sum_{m=1}^{i_0}\frac{d_0 m}{n}P(m,n)$$

$$=d_0\sum_{n=1}^{\infty}\sum_{m=1}^{i_0}\frac{m}{n}\{F_m(n)-F_m(n-1)\}\prod_{i=1}^{m-1}(1-F_i(n-1))\prod_{i=m+1}^{i_0}(1-F_i(n))$$

$$=d_0\sum_{n=1}^{\infty}\sum_{m=1}^{i_0}\frac{m}{n}\{F_m(n)-F_m(n-1)\}\exp\{\sum_{i=1}^{m-1}\ln(1-F_i(n-1))+\sum_{i=m+1}^{i_0}\ln(1-F_i(n))\}$$

$$(5\text{-}22)$$

Moreover, we approximate the summation over $m$ by integration, and let $i_0$ and $j_0$ approach infinity, so we have

$$\frac{da}{dn}=d_0\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}dm\ m\{F_m(n)-F_m(n-1)\}$$

$$\times\ \exp\{\int_0^m di\,\ln(1-F_i(n-1))+\int_m^{\infty}di\,\ln(1-F_i(n))\}. \qquad (5\text{-}23)$$

Similarly, from equation (5-21), the averarage of $N$, $\langle N\rangle$, is

$$< N > = \sum_{n=1}^{\infty} \exp\{ \sum_{i=1}^{i_0} \ln(1-F_i(n-1))\}$$

$$\simeq \sum_{n=1}^{\infty} \exp\{\int_0^{\infty} di \ln(1-F_i(n-1))\}. \qquad (5\text{-}24)$$

The significance of $<N>$ is given in section 5-B-5.

## 5-B-3 The Distribution Function of $H(i,j)$

In this section, $H(i,j)$, which expresses the life of element $(i,j)$, is investigated.

The damage $X(i,j)$ which is transiently stored under repeated stress in element $(i,j)$ per unit cycle is a random variable. It is assumed that $X(i,j)$ in each cycle is mutually independent and has the same distribution. For simplicity, we suppose that $X(i,j)$ is a random variable with an exponential distribution whose mean is $\sigma(i,j)$. That is, the distribution function of $X(i,j)$, denoted by $G_{ij}(x)$, is as follows:

$$G_{ij}(x) \equiv Pr.\{X(i,j) \leq x\} = \begin{cases} 1-\exp\{-x/\sigma(i,j)\} & \text{for} \quad x \geq 0 \\ 0 \quad . & \text{for} \quad x \leq 0 \end{cases} \tag{5-25}$$

Furthermore, we assume that only a part of the transiently stored damage beyond a certain amount $Q$ is accumulated in the form of dislocations, and that the element fails when the damage accumulated in it exceeds a threshold value $U$.

In order to obtain an expression for the life of each element, we suppose that no crack propagates, even if an element fails. In this situation, if the damage accumulated in one element does not exceed

$U$ in the $(n-1)$'th load cycle but does in the $n$'th load cycle, the life of this element is $n$ load cycles. Moreover, $U$ must be considered as a random variable due to the previous accumulation of damage. For simplicity, however, we assume that $U$ has a definite value. Consequently, we have

$$H_{ij}(h)-H_{ij}(h-1)$$
$$=\lambda e^{-\frac{U}{\sigma(i,j)}} \sum_{l=0}^{h-1} \frac{(h-1)!}{l!(h-l-1)!l!}\lambda^{l}(1-\lambda)^{h-l-1}\left(\frac{U}{\sigma(i,j)}\right)^{l}, \quad \text{for} \quad h=1,2,\cdots \tag{5-26}$$

where $\lambda=\exp\{-Q/\sigma(i,j)\}$, and

$$H_{ij}(h) = 0 \qquad \text{for} \quad h \leq 0 \tag{5-27}$$

The derivation of equation (5-26) can be performed in an analogous way to that in Appendix 4-1, in which $H_{ij}(h)-H_{ij}(h-1)$ is equal to $f(h)$ if, in equation (4-13), $w(S)$ and $\sigma$ are replaced by $\lambda$ and $\sigma(i,j)$, respectively. Equation (5-27) is evident from the definition. Thus $H_{ij}(h)$ for $h=1,2,\cdots$ is obtained from the following expression by the use of equations (5-26) and (5-27).

$$H_{ij}(h) = \sum_{l=1}^{h} (H_{ij}(l)-H_{ij}(l-1)) \tag{5-28}$$
$$h=1,2,\cdots$$

If we consider the $(i,j)$-dependence of $U$, we can consequently take into account the damage accumulation during the previous growth as in Part A. For simplicity we assume, however, that $U$ is the same constant for all elements. If $\sigma(i,j)$ is determined, we can obtain all information for the distribution of $H(i,j)$ necessary for calculating $da/dn$ and $<N>$.

As seen from the definition of region D, only a small amount of fatigue damage is accumulated outside it and the damage accumulation inside it is larger on the average near the crack tip. As $\sigma(i,j)$ is the average of the damage per cycle accumulated in element $(i,j)$, it is reasonable that $\sigma(i,j)$ is assumed as follows:

$$\sigma(i,j) = c_0 e^{-\frac{d_0}{\omega}i - \frac{d_0}{2\gamma\omega}|j|} , \tag{5-29}$$

where $\omega$ is the breadth of region D measured parallel to the x-axis, $2\nu\omega$ is one of region D measured parallel to the z-axis, and $c_0$ is a constant which represents the extent of damage per unit cycle. From equation (5-29), we can conclude that it is allowable for $i_0$ and $j_0$ to approach infinity in equations (5-23) and (5-24). Moreover, we assume that $\omega$ is proportional to the plastic zone size $\omega_p$. In small scale yielding $\omega_p$ can be expressed as

$$\omega_p = \kappa\left(\frac{\Delta K}{2S_y}\right)^2, \tag{5-30}$$

where $\kappa$ is a constant, $S_y$ dynamical yield stress and $\Delta K$ the stress intensity range. Substitutig equations (5-26)-(5-29) into equation (5-17), we calculate $F_i(y)$. Using it in equations (5-23) and (5-24), we can obtain $da/dn$ and $\langle N \rangle$ in terms of $\Delta K$.

## 5-B-4 A Comparison of Analytic and Experimental Results

At first, for simplicity in the estimation of $da/dn$, we choose $Q=0$. In this case, the damage accumulation occurs in every load cycle and there are two parameters, $U/c_0$ and $\nu$, in the expression for the relation between $\frac{1}{d_0}\cdot\frac{da}{dn}$ and $\frac{\omega}{d_0}$. The value of $\nu$ is taken to be $0.5$ because it is assumed that the breadth of region D measured parallel to the x-axis is nearly equal to the one measured parallel to the z-axis. The $\omega/d_0$-dependence of the non-dimensional crack propagation rate $\frac{1}{d_0}\cdot\frac{da}{dn}$ for some values of $U/c_0$ is represented in Fig.5-4. Figure 5-4 shows that $\frac{1}{d_0}\cdot\frac{da}{dn}$ is nearly proportional to some power of $\omega/d_0$ for each value of $U/c_0$. That is to say, it seems that the following relation is satisfied.

$$\frac{da}{dn} \propto \omega^{\alpha} \tag{5-31}$$

where $\alpha$ is a constant. Using the relation $\omega \propto (\Delta K)^2$, we have

$$\frac{da}{dn} \propto (\Delta K)^{2\alpha} \tag{5-32}$$

Therefore the present theory yields the power law for the fatigue crack propagation rate which is widely recognized experimentally. From Fig.5-4, the value of the exponent $2\alpha$ seems to be between $2$ and $4$, and decreases slightly as $U/c_0$ increases. Moreover, the value $\frac{1}{d_0}\cdot\frac{da}{dn}$ for a fixed value of $\omega/d_0$ decreases as $U/c_0$ increases. Although we do not show the $\nu$-dependence of $da/dn$, $da/dn$ is not so sensitive to the value of $\nu$ in our calculation. So hereafter we take $\nu$ to be $0.5$.

In Fig.5-5, we show the experimental results for the through crack propagation by the use of low-carbon steel S20C with three different grain sizes[21]. The crack propagation rate for $\Delta K$ near $\Delta K_{th}$ is plotted as a function of $\Delta K_{eff}$ (effective stress intensity range). In Fig. 5-5, our theoretical $da/dn$ for $U/c_0=10.0$, $\nu=0.5$, $Q=0$ and $d_0=10^{-9}$m is shown to be in general agreement with the experiments. The value of $\omega/\omega_p$ is about $1/100$ for three kinds of materials.

Next, we investigate $da/dn$ for the case of $Q\neq 0$. In Fig.5-6, $\frac{1}{d_0}\cdot\frac{da}{dn}$ is plotted as a function of $\omega/d_0$ for $U/Q=3.125$ and $U/Q=1.0$. For $Q\neq 0$, the power law for $da/dn$ is also nearly satisfied for various values of $U/c_0$. Furthermore, we obtain $2\alpha=4.85$ for $U/c_0=4.0$ in Fig.5-6(b). In the case of $Q=0$, $2\alpha$ is less than $4$ in our calculation, so that $2\alpha$ larger than $4$ may not be obtained unless we take into account the effect of $Q$, i.e. a part of the transiently stored damage beyond a certain amount $Q$ is accumulated. The value of $\alpha$ becomes larger with a decrease in the value of $U/Q$.

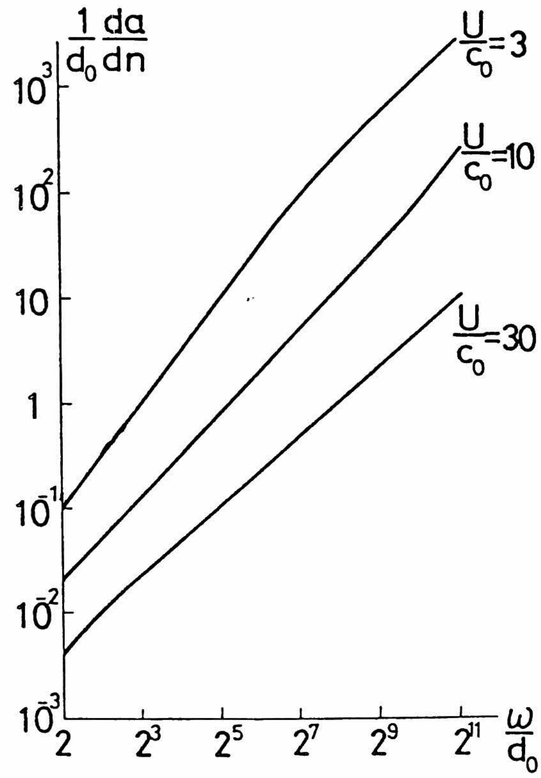


Fig. 5-4 Fatigue crack propagation rate $\frac{1}{d_0}\cdot\frac{da}{dn}$ as a function of $\omega/d_0$ for some values of $U/c_0$ according to our theory ($Q=0$).

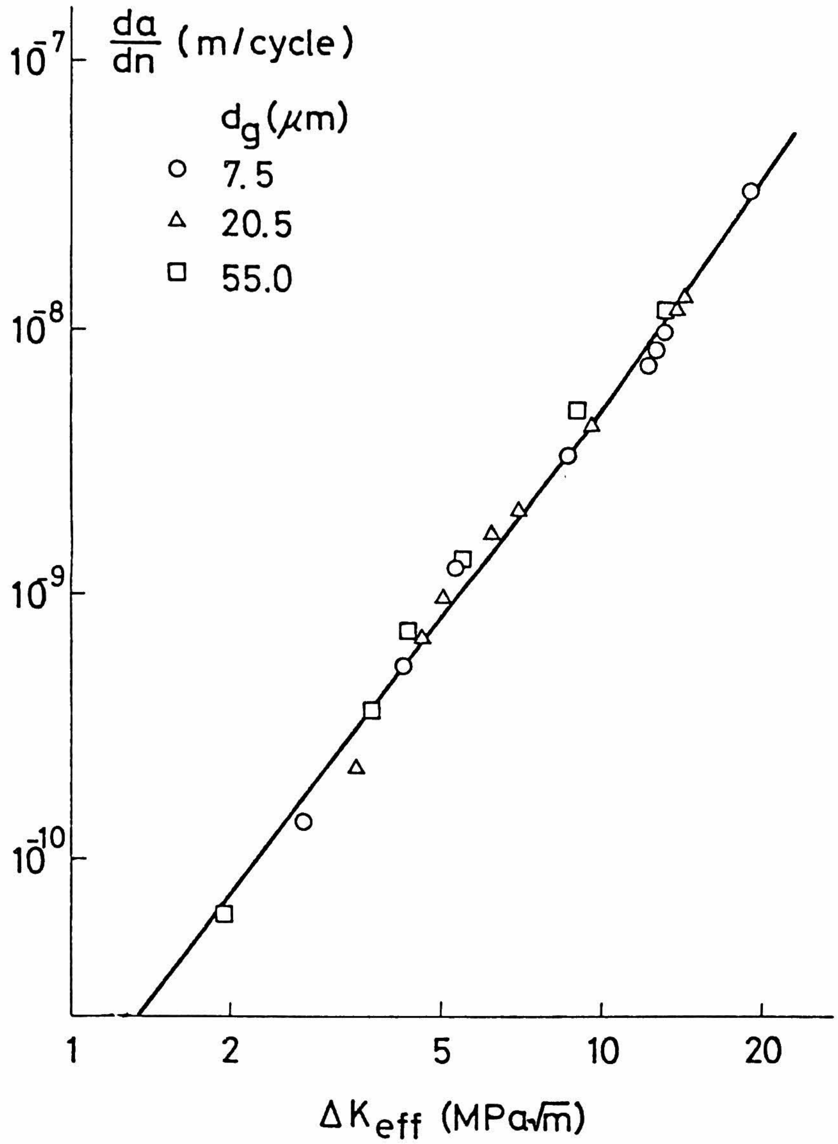


Fig. 5-5 Fatigue crack propagation rate *da/dn* as a function of effective stress intensity range $\Delta K_{eff}$ according to our theory (full curve) compared with experimental results (○,△,□)[21]. $d_g$ denotes the grain size.

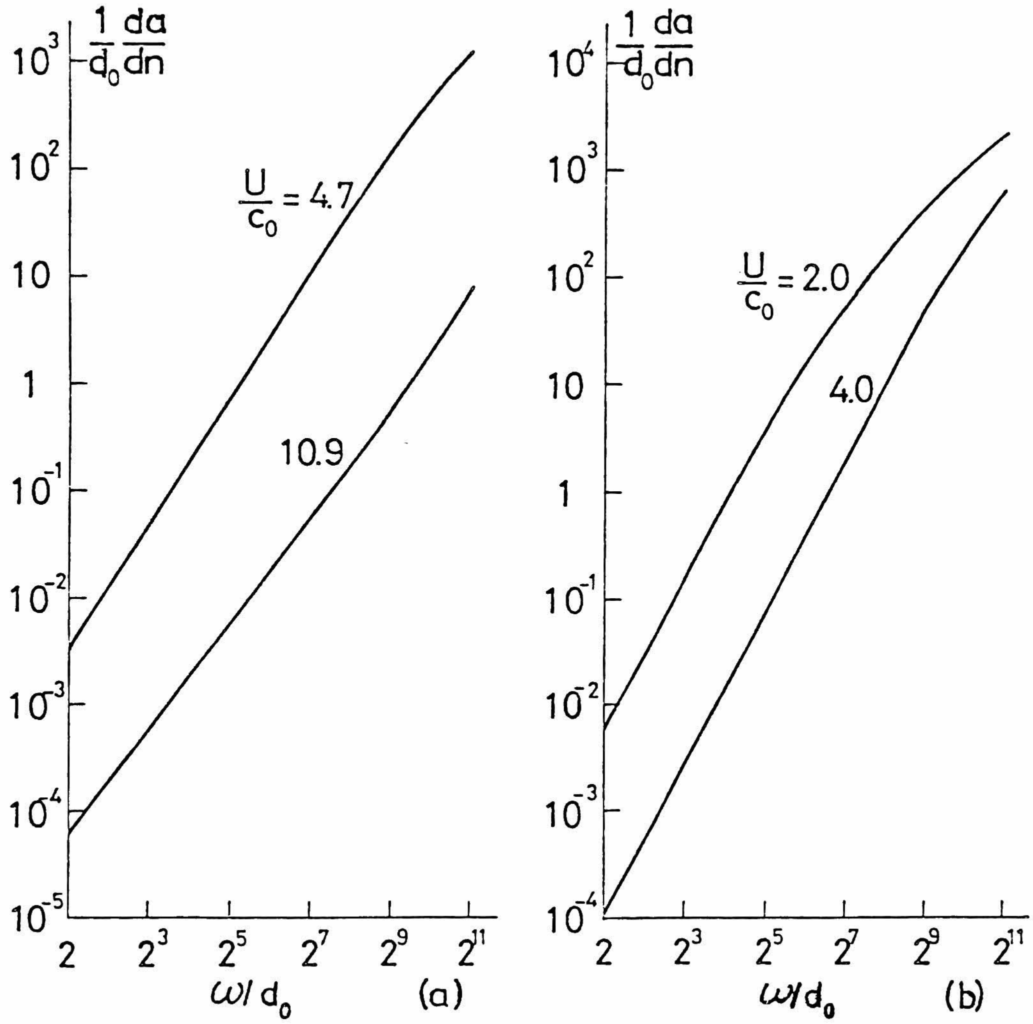


Fig. 5-6 Fatigue crack propagation rate $\frac{1}{d_0}\cdot\frac{da}{dn}$ as a function of $\omega/d_0$ according to our theory.

(a) $U/Q=3.125$, $\nu=0.5$.

(b) $U/Q=1.0$, $\nu=0.5$.

5-B-5 Some Further Remarks

The plastic zone size $\omega_p$ is generally given by equation (5-30) but the value of crack opening stress is not zero due to residual stress, so $\omega_p$ is related to $\Delta K_{eff}$ rather than $\Delta K$ or $\Delta K_{max}$. Consequently, the results according to our theory represent the relation between $da/dn$ and $\Delta K_{eff}$, and they are in good agreement with the results from the experiments.

The small part of the material, named region D, where the damage accumulation is remarkable, seems to be the highly heterogeneous region near the crack tip (called end region or process region[60]). In that region, the microstructure of materials is extremely disordered, so that the elements in the region may frequently fail due to damage accumulation. As it is concluded that $\omega/\omega_p$ is about *1/100* from the comparison with experimental results shown in Fig.5-5, it seems that region D is much smaller than the plastic zone.

From equation (5-24), we can calculate the average load cycle $\langle N \rangle$ expended for one step of crack growth. It is about *5* for $da/dn \simeq 10^{-9}$ m/cycle which is the value of $da/dn$ for $\Delta K$ slightly larger than $\Delta K_{th}$ ( threshold stress intensity range) and it decreases to *1* with an increase of $\Delta K$. From this fact, it may be understood that our model is quite consistent with actual fatigue crack growth phenomena.

The threshold value $U$ of damage accumulation is essentially a random variable, although we give it a definite value here in Part B. We hope to treat it as a random variable in the near future.

# CHAPTER 6

# CONCLUDING REMARKS

Here, we give some remarks on the contents of the preceding chapters.

In this thesis we investigated fatigue of metals on the basis of a stochastic model for dislocation multiplication. In chapter 2 we investigated crack initiation in high-cycle fatigue and proposed a propagation law of small fatigue cracks in chapter 3. In chapter 4 we improved the damage accumulation model which is used in studying the high-cycle fatigue phenomena in order that elasto-plastic fatigue can be treated. In chapter 5 a law of through fatigue crack propagation was investigated on the basis of a kind of weakest link model. Furthermore, we compared our numerical results with experimental data, using the damage accumulation model in calculating the distribution function of lives of material elements. From the foregoing argument presented in the preceding chapters, we can conclude by saying that fatigue phenomena of metals can be treated by the use of the stochastic model of dislocation multiplication, and that various aspects of fatigue phenomena can be explained with this model.

In Stage I, we assume that the life of each cell from the beginning of damage accumulation up to failure is a mutually independent random variable with the same distribution. In practice, however, the life of each cell cannot be considered to be a random variable with the same distribution, because the crack lengths vary different when each cell fails. We hope to treat Stage I using a more realistic model in the near future.

In chapter 3 we showed the propagation rate of fatigue cracks whose length is of the order of grain size is proportional to $J^{(N)}$ (fatigue J-integral). The explanation of this law according to the consideration of microscopic mechanism is the next subject.

Furthermore, we expect that fatigue in elevated temperature and creep-fatigue interaction can be treated by the use of a model similar to the one proposed in this thesis. By using the damage accumulation model, we hope and believe that these phenomena will be fully understood in the near future.

REFERENCES


[1] W.A.J.Albert, Arch.Miner.Geognosie Berg Hüttenkunde, 10(1838),215.

[2] W.J.M.Rankine, Proc. Instn.Civ.Engrs., 2(1843),105.

[3] E.A.Hodgkinson, H.M.S.O.Command Paper No.1123 (1849).

[4] W.Foirbairn, Phil.Trans.R.Soc., 154(1864),311.

[5] A.Wöhler, Z.Bauw, 8(1854),642;10(1860),583;13(1863), 233;16(1866),67;20(1870),74; Engineering, 11(1871),199.

[6] See, for instance,
T.Kawamoto, J.Soc.Mat.Sci.,Japan,(in Japanese) 25 (1976),810.
N.E.Frost, K.J.Marsh and L.P.Pook,"Metal Fatigue", (Clarendon Press,1974).

[7] M.H.Raymond and L.F.Coffin, ASME J.Bas.Eng., 85(1963), 548.

[8] P.J.E.Forsyth, Crack Propagation Symposium, Cranfield, (1961),76.

[9] J.A.Ewing and J.C.W.Humphrey, Phil.Trans., A200(1903), 241.

[10] N.Thompson, International Conference on Fatigue, Institution on Mechanical Engineers, (1956),527; "Fracture",(Technology Press,Wiley 1959) pp.354.

[11] G.Birkgeck, A.E.Inckle and G.W.J.Waldron,J., Mat.Sci., 6(1971),319.

[12] H.Nakazawa and H.Kobayashi,J.Jap.Soc.Mech.Eng.,(in Japanese) 75(1972),1118.

[13] P.C.Paris, Ph.D.Thesis Lehigh University, (1962).

[14] P.C.Paris, M.P.Gomez and W.E.Anderson, The Trend in Engineering, 13(1961),9.

[15] See, for instance,
D.Broek,"Elementary Engineering Fracture Mechanics", (Sijthoff & Noordhoff,1978).

[16] D.Broek and J.Schijve, Aircraft Engineering, 39(1967), 10.

[17] D.P.Wilhem, ASTM STP 415(1967),363.

[18] H.Kitagawa and C.M.Suh, Preprint of 13th Symp. on Fatigue, Jap.Soc.Mat.Sci.,(in Japanese) (1979),10.

[19] K.Tanaka, Z.Nakai, M.Hojo, H.Maekawa and T.Hoshide, Preprint of 13th Symp. on Fatigue, Jap.Soc.Mat.Sci., (in Japanese) (1979),1.

[20] S.J.Hudak,Jr, ASME J.Eng.Mat.Tech., 103(1981),26.

[21] S.Taira, K.Tanaka and M.Hoshina, ASTM STP 675(1979), 135.

[22] T.Yokobori,"Strength, Fracture and Fatigue of Metals", (Clarendon Press,1974).

[23] C.Ihara and A.Igarashi, J.Soc.Mat.Sci.,Japan, (in Japanese) 29(1980),434.

[24] C.Ihara and A.Igarashi, J.Soc.Mat.Sci.,Japan, (in Japanese) 30(1981),273.

[25] A.Igarashi, J.Soc.Mat.Sci.,Japan, (in Japanese) 31 (1982),574.

[26] K.P.Oh, ASME J.Eng.Mat.Tech., 100(1978),170.

[27] A.Tsurui and A.Igarashi, ASME J.Eng.Mat.Tech., 102 (1980),300.

[28] C.Ihara and A.Igarashi, J.Soc.Mat.Sci.,Japan, (in Japanese) 30(1981),574.

[29] C.Ihara and A.Igarashi, ASME J.Eng.Mat.Tech, 103 (1981),293.

[30] See, for instance,
Preprints of 13th Symp. on Fatigue, Jap.Soc.Mat.Sci., (in Japanese) (1979).
Preprints of 16th Symp. on Fatigue Strength, Jap.Soc. Mech.Eng., No.790-9, (in Japanese) (1979).

[31] S.Taira, K.Tanaka and M.Hoshina, ASTM ATP 675(1979), 135.

[32] C.Ihara and A.Tsurui, ASME J.Eng.Mat.Tech., 99(1977), 26.

[33] J.Piqueras, J.C.Grosskreutz and W.Frank, Phys.Stat. Sol., 11(1972),567.

[34] H.Nakazawa, J.Soc.Mat.Sci.,Japan, (in Japanese) 26 (1977),99.

[35] J.Schlipf, W.Winkler-Gniewek and R.Schindlmayr, Phil.Mag.A, 41(1980),509.

[36] J.E.Pratt, Acta Met., 15(1967),319.

[37] G.Langford and M.Cohen, Trans.Am.Soc.Metals, 62(1969), 623.

[38] C.Ihara, A.Tsurui and A.Igarashi, ASME J.Eng.Mat.Tech., 99(1977),212.

[39] C.E.Feltner and J.Morrow, ASME J.Bas.Eng., 83(1961), 15.

[40] A.Tsurui and S.Osaki, Stoch.Process their Appl., 4 (1976),79.

[41] S.Taira, R.Ohtani and T.Kitamura, ASME J.Eng.Mat.Tech., 101(1979),154;
S.Taira, R.Ohtani and T.Komatsu, ibid. 101(1979),162;
R.Koterazawa and T.Mori, ibid. 99(1977),298;
K.Ohji, K.Ogura, S.Kubo, Y.Katada and T.Katsuhara, Trans.Jap.Soc.Mech.Eng., (in Japanese) A45(1979),550.

[42] C.Ihara, J.Soc.Mat.Sci.,Japan, (in Japanese) 30(1981), 158.

[43] C.F.Shih and J.W.Hutchinson, ASME J.Eng.Mat.Tech., 98(1976),289.

[44] K.Ohji, K.Ogura, S.Kubo, Trans.Jap.Soc.Mech.Eng., 42 (1976),350;43(1977),1577;(in Japanese)
J.D.Landes and J.A.Begleg, ASTM STP 590(1976),128.

[45] A.Tsurui, Proc.23rd Japan Congr.Mat.Research (1980), 18.

[46] E.Shiratori and Y.Obataya, Trans.Jap.Soc.Mech.Eng., (in Japanese) 35(1969),711.

[47] K.Tanaka, T.Hoshide and O.Maekawa, Eng.Frac.Mech., 16(1982),207.

[48] J.D.Eshelby,"Inelastic Behavior of Solids", edited by M.F.Kanninen et.al. (McGrow-Hill, New York, 1970).

[49] J.R.Rice, ASME J.Appl.Mech., 35(1968),379.

[50] J.C.Amazigo, Int.J.Solids Struc., 10(1974),1003.

[51] T.Yamada, T.Hoshide and M.Manabe, Preprint of 20th Symp. on Fracture of Materials and Structures, Jap. Soc.Mech.Eng., No.810-7, (in Japanese) (1981),303.

[52] See, for instance, R.O.Ritchie, ASME J.Eng.Mat.Tech., 99(1977),195.

[53] R.W.Lardner, Phil.Mag., 17(1968),71.

[54] B.Tomkins, Phil.Mag., 18(1968),1041.

[55] J.Weertman, Int.J.Frac.Mech., 2(1966),460.

[56] S.Majumdar and J.Morrow, ASTN STP 559(1974),159.

[57] T.Nakazawa and S.Osaki, IEEE Trans. on Reliability R-24(1975),300.

[58] G.G.Garrett and J.F.Knott, Met.Trans., 7A(1976),884.

[59] K.Tanaka, T.Hoshide and S.Taira, J.Soc.Mat.Sci.,Japan, (in Japanese) 28(1979),41.

[60] K.B.Broberg, J.Mech.Phys.Sol., 19(1971),407.

PUBLICATION LIST


1. C.Ihara, A.Tsurui and A.Igarashi,"Fatigue Lives of Metals up to Crack Initiation," ASME J.Eng.Mat.Tech.,99(1977),212.

2. T.Munakata and A.Igarashi,"Dynamical Correlation Functions of Classical Liquids.I," Prog.Theor.Phys.,58(1977),1345.

3. T.Munakata and A.Igarashi,"Dynamical Correlation Functions of Classical Liquids.II—A Self-Consistent Approach," Prog. Theor.Phys.,60(1978),45.

4.* C.Ihara and A.Igarashi,"A Crack Initiation Model in High-Cycle Fatigue of Low-Carbon Steel," J.Soc.Mat.Sci.,Japan, 29(1980),434. (in Japanese)

5.* A.Tsurui and A.Igarashi,"A Probabilistic Approach to Fatigue Crack Growth Rate," ASME J.Eng.Mat.Tech.,102 (1980),300.

6.* C.Ihara and A.Igarashi,"A Propagation Law of Small Fatigue Cracks," J.Soc.Mat.Sci.,Japan,30(1981),273. (in Japanese)

7.* C.Ihara and A.Igarashi,"A Law of Fatigue Crack Propagation on the Basis of a Stochastic Model for Damage Accumulation," J.Soc.Mat.Sci.,Japan,30(1981),574. (in Japanese)

8.* C.Ihara and A.Igarashi,"A Damage Accumulation Model for Fatigue Crack Propagation," ASME J.Eng.Mat.Tech.,103 (1981),293.

9.* A.Igarashi,"A Damage Accumulation Model for Elasto-Plastic Fatigue," J.Soc.Mat.Sci.,Japan,31(1982),574. (in Japanese)

* The papers on which the present thesis is based.